The Equivariant *K*-localization of the *G*-Sphere Spectrum

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The two other parts are

- Oriented, Equivariant *K*-theory and the Sullivan Splitting
- The *K*-localizations of Some Classifying Spaces

The Equivariant *K*-localization of the *G*-Sphere pectrum

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The purpose of this paper is to generalize Bousfield's calculation in [B79b], (4.3) of the *K*-localization of the sphere spectrum to the equivariant case.

This is done as follows: We select an odd prime *p*, and then consider two different cases:

I: *G* is a *p*-group, and

II: *G* is a finite group with order prime to *p*.

In both cases we work *p*-locally.

In section I we describe the *G* Spaces $Q_G S^0$ and $K(\mathbb{F}_q, G)$. In section 2 we consider the two different cases above and define the relevant infinite *G*-loop space, $J(G,p)$, which is to give the K_G-localization of the *G*-sphere spectrum S_G. We also define the infinite *G*-loop map $e(G,p): Q_G S^0 \to J(G,p)$, and the *G*-space $\operatorname{Cok} J(G,p)$ as the homotopy fibre of $e(G,p)$.

In section 3 we show that in case II we have a splitting

$$SF_G \simeq J(G,p)_0 \times \operatorname{Cok} J(G,p)_0,$$

where SF_G, $J(G,p)_0$ and $\operatorname{Cok} J(G,p)_0$ denote the *G*-connected covers of $Q_G S^0$, $J(G,p)$ and $\operatorname{Cok} J(G,p)$, respectively. At the moment it doesn't seem to be possible to prove an analogous statement in case I.

In section 4 we study the K_G-theory of S_G and of $J(G,p)$.

In section 5 we briefly describe the properties of equivariant Bousfield-localization with respect to a *G*-spectrum, and we define the $\mathcal{K}_G$-localization, which in a certain sense is the correct localization to use. A *G*-spectrum *X* is $\mathcal{K}_G$-local, if and only if for every subgroup *H* of *G* the fixed point spectrum X^H is *K*-local.

Finally, in section 6 we calculate the $\mathcal{K}_G$-localization of S_G.

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1. Preparations

In this section we collect results to be used in the following. We start by defining the spaces $Q_G S^0$, SF_G and $K(\mathbb{F}_q, G)$:

Definition 1.1

Let $Q_G S^0$ be the G-loop-space $\varinjlim \Omega^V S^V$, where the limit is over all *G*-modules in a fixed *G*-universe $\mathcal{U}$, cf. [LMS], p.l l ff. we demand that $\mathcal{U}$ is a complete *G*-universe, i.e. that $\mathcal{U}$ contains countably many copies of the regular representation $\mathbb{R}[G]$.

Let SF_G be the *G*-connected cover of $Q_G S^0$, cf [El], p.277. We here let the basepoint of $Q_G S^0$ be the identity map between *G*-spheres.

Proposition 1.2 ([S70], p.62)

$(Q_G S^0)^G \simeq \prod_{(H)} Q(BW_G H_+)$ and $(SF_G)^G \simeq \prod_{(H)} Q_0(BW_G H_+)$, where the product is over all conjugacy classes (H) of subgroups of *G*, and $W_G H$ is the Weyl-group $N_G(H)/H$. $Q_0(BW_G H_+)$ is the basepoint component of $Q(BW_G H_+)$.

Remark 1.3

It is well known, [Sh], p.242, that the infinite *G*-loop-space $Q_G S^0$ can alternatively be obtained as follows:

Let $\mathcal{E}_G$ be the category, whose objects are pairs (n,ρ), where *n* is a non-negative integer, and $\rho: G \to \Sigma_n$ is a homomorphism. The set of morphisms $(m,\rho) \to (n,\tau)$ is empty if $m \neq n$, and the set of all bijections $\{1,...,m\} \to \{1,...,n\}$ if $m = n$.

G acts on $\mathcal{E}_G$ as follows: *G* acts trivially on objects, and if $f:(m,\rho) \to (n,\tau)$ is a morphism, and $g \in G$, then gf is the morphism sending $i \in \{1,...,m\}$ to $\tau(g)(f((\rho^{-1}(g)(i))))$.

$\mathcal{E}_G$ has a composition given by the disjoint union $\coprod$, $(m,\rho)\coprod(n,\tau) = (m+n, \rho\coprod\tau)$, where $\rho\coprod\tau : G \to \Sigma_{m+n}$ is the composite $G \xrightarrow{\rho\times\tau} \Sigma_m \times \Sigma_n \longrightarrow \Sigma_{m+n}$.

This makes $\mathcal{E}_G$ into a permutative *G*-category, and according to [Sh], thm. A', p. 255, the group completion $\Omega B(B\mathcal{E}_G)$ of $\mathcal{E}_G$ is an infinite *G*-loop-space. It follows from [Sh], p.242, that $\Omega B(B\mathcal{E}_G)$ is *G*-homotopy equivalent to $Q_G S^0$ as an infinite *G*-loop-space.

Definition 1.4 ([FHM], (2.1))

Let *q* be a prime power. Let $\mathcal{GL}_G(\mathbb{F}_q)$ be the category, whose objects are pairs (n,ρ), where *n* is a non negative integer, and $\rho: G \to Gl_n(\mathbb{F}_q)$ is a homomorphism. The set of morphisms $(m,\rho) \to (n,\tau)$ is empty if $m \neq n$, and is the set of all $\mathbb{F}_q$-isomorphisms $\mathbb{F}_q^{\ m} \to \mathbb{F}_q^{\ n}$ when $m = n$.

G acts on $\mathcal{GL}_G(\mathbb{F}_q)$ like the *G*-action on $\mathcal{E}_G$, and direct sum of $\mathbb{F}_q$-modules makes $\mathcal{GL}_G(\mathbb{F}_q)$ into a permutative *G*-category.

The corresponding infinite *G*-loop-space $\Omega B(B\mathcal{GL}_G(\mathbb{F}_q))$ is denoted $K(\mathbb{F}_q, G)$ – see e.g. [Sh], p.242, or [FHM], (0.2).

Proposition 1.5 ([FHM], (3.1))

Assume $(q,|G|) = 1$. Let $V_1, V_2, ..., V_n$ be the irreducible $\mathbb{F}_q G$-modules, and let $D_i = \mathrm{Hom}_{\mathbb{F}_q G}(V_i, V_i)$ be the corresponding finite field. Then

$$(K(\mathbb{F}_q, G))^G \simeq \prod_{i=1}^{n} K(D_i) \simeq \prod_{i=1}^{n} K(\mathbb{F}_q[G])$$

Definition 1.6

The functor $P:\mathcal{E}_G \to \mathcal{GL}_G(\mathbb{F}_q)$ is defined as follows. *P* maps rhe object $(n,\rho:G\to\Sigma_n)$ of $\mathcal{E}_G$ to $(n,\rho:G\to Gl_n(\mathbb{F}_q))$, where Σ_n is embedded in $Gl_n(\mathbb{F}_q)$ as the subgroup permiting the standard basis. On morphisms we let $P(f:(m,\rho)\to(n,\tau))$ the map $P(f):\mathbb{F}_q{}^m\to\mathbb{F}_q{}^n$ mapping the *i*'th standard basis vector e_i to $e_{f(i)}$.

P is seen to preserve the *G* action and the permutative structure.

Definition 1.7

The infinite *G*-loop-map $e_G:Q_G S^0\to K(\mathbb{F}_q,G)$ is defined to be

$$e_G=\Omega B(B(P)):Q_G S^0=\Omega B(B\mathcal{E}_G)\to\Omega B(B\mathcal{GL}_n(\mathbb{F}_q))=K(\mathbb{F}_q,G)$$

We use the notation of [El] and let $\mathcal{O}_G$ denote the category of *G*-orbits, i.e. the objects of $\mathcal{O}_G$ are the transitive *G*-sets G/H, where *H* ranges over all the subgroups of *G*. The morphisms are all *G*-maps.

For a *G*-space *X*, we let ΦX denote the functor $\mathcal{O}_G\to\{\text{pointed topological spaces}\}$ given by

$$(\Phi X)(G/H)=X^H.$$

(In general, a functor $\mathcal{O}_G\to\{\text{pointed topological spaces}\}$ is denoted an $\mathcal{O}_G$-space). Φ is seen to be a functor from {*G*-spaces} to {$\mathcal{O}_G$-spaces}.

We furthermore have the functor C: {$\mathcal{O}_G$-spaces} $\to$ {*G*-spaces}, which is the right adjoint to Φ in the corresponding homotopy categories, i.e. we have the natural bijection

(1.8) $\quad [X,CT]^G\cong[\Phi X,T]_{\mathcal{O}_G}$ [X,CTIG N (ØX,Tloc

of [El], thm. 2, where *X* is a *G*-space and *T* an $\mathcal{O}_G$-space.

We describe in detail the $\mathcal{O}_G$-space $\Phi(Q_G S^0)$:

By considering the category $\mathcal{E}_G$ of (1.3), we see that $(\mathcal{E}_G)^H$ is equivalent to the category $\mathcal{S}_H$ consisting of all finite *H*-sets and *H*-equivalences. $\mathcal{S}_H$ splits as a category into factors $\mathcal{T}_{H/K}$, where H/K is the typical irreducible *H*-set, and where the product ranges over all the *H*-conjugacy classes $(K)_H$ of supgroups *K* of *H*. $\mathcal{T}_{H/K}$ is the full subcategory of $\mathcal{S}_H$ consisting of the objects $n(H/K), n\geq 0$.

From (1.2) we have that

$$(Q_G S^0)^H\simeq(Q_H S^0)^H\simeq\prod_{(K)_H}Q(BW_H K_+)$$

where the product runs over all *H*-conjugacy classes of subgroups *K* of *H*. In view of the discussion above, we see that the factor $Q(BW_H K_+)$ originates as $Q(BAut_H(H/K)_+)$ – recall that $Aut_H(H/K) = W_H K$.

Let $K \le H \le G$ be subgroups. The projection map $G/K \to G/H$, which is a morphism of $\mathcal{O}_G$, induces the inclusion $(Q_G S^0)^H \to (Q_G S^0)^K$. This map is described as follows:

Let $S_1 = H/A_1$, $S_2 = H/A_2$, ..., $S_n = H/A_n$, and $T_1 = K/B_1$, $T_2 = K/B_2$ ... , $T_m = K/B_m$ be a complete list of the inequivalent, irreducible *H*-sets and *K*sets, respectively. The integers a_{ij} , $1 \le i \le m$, $1 \le j \le n$, are defined by

$$Res_K^H(S_j) \cong \coprod_{i=1}^m a_{ij} T_i$$

Furthermore, we have the group homomorphism

$$r_{ij} = W_H A_j = Aut_H(S_j) \to Aut_K(a_{ij} T_i) = \Sigma_{a_{ij}} \int W_K B_i$$

well defined up to inner automorphisms. r_{ij} gives a functor $\mathcal{T}_{H/A_j} \to \mathcal{T}_{K/B_i}$, and we obtain an infinite loop map $R_{ij} : Q(BW_H A_j) \to Q(BW_K B_i)$.

The map $S : (Q_G S^0)^H = \prod_{j=1}^m Q(BW_H A_j) \to \prod_{i=1}^n Q(BW_K B_i) = (Q_G S^0)^K$ is now given by the $m \times n$ matrix:

$$(1.9) \qquad A_K^H = \begin{pmatrix} a_{11}R_{11} & a_{12}R_{12} & \cdots & a_{1n}R_{1n} \\ a_{21}R_{21} & a_{22}R_{22} & \cdots & a_{2n}R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}R_{m1} & a_{m2}R_{m2} & \cdots & a_{mn}R_{mn} \end{pmatrix}$$

This is seen as follows: As both *S* and A_K^H are infinite loop maps, they are determined up to homotopy by their restrictions to $\coprod_{j=1}^m BW_H A_j$. These restrictions coincide, and we conclude that *S* and A_K^H are homotopic maps.

A general morphism $f : G/K \to G/H$ in $\mathcal{O}_G$ is of the form $f(gH) = gaK$, where $a \in G$ is given by $f(H) = aK$. It is seen that $a^{-1}Ka \le H$, and as there is a one to one correspondance between $a^{-1}Ka$ -sets and *K*-sets, the induced map $\Phi(Q_G S^0)(f)$ is given by $I_K^{a^{-1}Ka} \circ A_{a^{-1}Ka}^H$, where $I_K^{a^{-1}Ka}$ is the $m \times m$ -matrix given as follows: Let the irreducible *K* sets be $\{T_1, T_2, ..., T_m\}$, and let the irreducible $a^{-1}Ka$ -sets be $\{U_1, ..., U_m\}$. Then the (i, j) 'th entry of $I_K^{a^{-1}Ka}$ is 1 if T_i corresponds to U_j under the 1-1 correspondance above, and is zero otherwise.

2. Equivariant *J*-theory

In this section we define equivariant *J*-theory. We fix an odd prime *p*, and we define the *p*-local, equivariant space $J(G, p)$ in two cases:

I: when G is a p-group, and
II: when G is a finite group with order $|G|$ relatively prime to p.

In both cases we select a prime q, such that $q+p^2\mathbb{Z}$ generates the unit group $(\mathbb{Z}/p^2)^\times$, and such that $(q,\|G\|)=1$. Such an integer q exists according to Dirichlet's theorem, cf. [Ap], (7.9).

The next key definition is inspired of [MR89], thm. C:

<u>Definition 2.1</u>

I: Let G be a p-group. Define the functor

$$F : \{G\text{-}CW\text{-complexes}\} \to \{\text{Abelian groups}\}$$

by

$$F(X) = [X^G; K(\mathbb{F}_q)] \times \prod_{\substack{(H)\\ H\neq G}} [X^H; K(\mathbb{F}_q)\times K(\mathbb{F}_q, W_G H)]^{W_G H}$$

where the product is over all conjugacy elasses (H) of proper subgroups H of G, where we recall that X^H is canonically a $W_G H$-space, and where $W_G H$ acts trivially on $K(\mathbb{F}_q)$.

For the sake of simplicity we denote by $L(H)$ the $W_G H$-space $K(\mathbb{F}_q)\times K(\mathbb{F}_q, W_G H)$, when $H<G$, and the G-space $K(\mathbb{F}_q)$, when $H=G$.

Thus,

$$F(X) = \prod_{(H)} [X^H; L(H)]^{W_G H}$$

(This notation will normally only be used in proofs.)

II: Let G be a finite group of order prime to p. Define the functor

$$F : \{G\text{-}CW\text{-complexes}\} \to \{\text{Abelian groups}\}$$

by

$$F(X) = \prod_{(H)} [X^H; K(\mathbb{F}_q)]^{W_G H}$$

where the product is over all conjugacy classes (H) of subgroups H of G, and where $W_G H$ acts trivially on $K(\mathbb{F}_q)$.

<u>Definition 2.2</u>

F satisfies the requirements of the equivariant Brown representation theorem, [LMS], (1.5.11), in both case I and II. We thus get a G-space $J(G,p)$ representing F, i.e.

(2.3) $\quad F(X) \cong [X, J(G,p)]^G$,

for every finite, based G-CW-complex X. (See also [B65], thm. (2.8), the proof of which carries over to the equivariant case without complications.)

Proposition 2.4

Let *H* be a subgroup of *G*. Then

$$J(G,p)^H \simeq K(\mathbb{F}_q)\times\prod_{\substack{(K)_H\\ K\neq H}} K(\mathbb{F}_q)\times K(\mathbb{F}_q[W_H K]),$$

if we are in case I (i.e. if *G* is a *p*-group), and

$$J(G,p)^H \simeq \prod_{(K)_H} K(\mathbb{F}_q)$$

in case II. In both cases the product runs over *H*-conjugacy classes $(K)_H$ of subgroups *K* of *H*.

Proof:

We only prove the statement in case I, as the proof in case II is virtually unchanged.

As $[X,Y^H]\cong[X\wedge(G/H_+),Y]^G$ for a (non-equivariant) space *X*, a *G*-space *Y* and a subgroup *H* of *G*, we get the following calculations:

$$[X,J(G,p)^H]\cong[X\wedge(G/H_+),J(G,p)]^G\cong\prod_{(K)_G}[X\wedge(G/H_+),L(K)]^{W_G K}$$

$(G/H)^K=\varnothing$, if *K* is not conjugate to a subgroup of *H*, while

(2.5) $$(G/H)^K=\coprod_{i=1}^{n} N_G(K_i)/(N_G(H_i)\cap H)$$

as a $W_G H$ -set, if *K* is conjugate to a subgroup of *H*:

It suffices to consider the case where *K* is a subgroup of *H*. Let $K_1,K_2,...,K_n$ be a full collection of *H*-conjugacy classes of subgroups of *H*, such that K_i is *G*-conjugate to *K*. As $(G/H)^K=\{gH\in G/H\mid gKg^{-1}\leq H\}$, we study the set $G_*=\{g\in G\mid gKg^{-1}\leq H\}$. $G_*=G_*^1\coprod G_*^2\coprod...\coprod G_*^n$, where $G_*^i=\{g\in G\mid gKg^{-1}$ is H-conjugate to $K_i\}$. It is easily seen that the $W_G H$ -set G_*^i/H is isomorphic to $N_G(K_i)/(N_G(K_i)\cap H)$, thus proving (2.5).

We now see that

$$[X,J(G,p)^H]\cong\prod_{(K)_H}[X\wedge(H_G(K)/(N_G(K)\cap H)_+):L(K)]^{W_G K}\cong$$

$$\prod_{(K)_H}[X\wedge((H_G(K)/K)/(N_H(K)/K)_+):L(K)]^{W_G K}\cong$$

$$\prod_{(K)_H}[X;L(K)^{W_H K}]\cong[X;K(\mathbb{F}_q)]\times\prod_{\substack{(K)_H\\ K\neq H}}[X;K(\mathbb{F}_q)\times K(\mathbb{F}_q[W_H K])]$$

as it follows from (1.5). This proves the theorem.

QED

Remark 2.6

If $K\leq H\leq G$, and we are in case II, the inclusion

$$\prod_{j=1}^{m}K(\mathbb{F}_q)\simeq J(G,p)^H\to J(G,p)^K\simeq\prod_{j=1}^{n}K(\mathbb{F}_q)$$

is given by the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Here the a_{ij}'s are defined as in (1.9).

This follows from the proof of (2.4): Let $\pi : G/K \to G/H$ be the projection, and let $i : J(G,p)^H \to J(G,p)^K$ be the inclusion.

For a *CW*-complex *X* we now see that the diagram

$$\begin{array}{ccc} [X, J(G,p)^H] & \stackrel{\sim}{\to} & [X \wedge (G/H_+); J(G,p)]^G \\ \downarrow i_* & & \downarrow (Id_X \wedge \pi)^* \\ [X, J(G,p)^K] & \stackrel{\sim}{\to} & [X \wedge (G/K_+); J(G,p)]^G \end{array}$$

commutes. By using the fact that the splittings of $J(G,p)^H$ and $J(G,p)^K$ come from the various irreducible *H*- and *K*-sets, we get the result.

<u>Proposition 2.7</u>

$J(G,p)$ is an infinite *G*-loop-space, both in case I and in case II.

<u>Proof:</u>

Again we only prove the statement in case I.

From (1.4) it follows that $L(H)$ is an infinite $W_G H$-loop space. Thus, for every $W_G H$-representation U_H we have an U_H'th delooping $L(H)_{U_H}$, such that $\Omega^{U_H}(L(H)_{U_H})$ and $L(H)$ are *G*-homotopy equivalent *G*-spaces.

Let *V* be a *G*-module. Then the *V*'th delooping of $J(G,p)$ is the representing space for the functor

$$F_V : X \to \prod_{(H)} [X^H ; L(H)_{V^H}]^{W_G H}$$

where the fixed point representation V^H is considered as a $W_G H$-module: Let BF_V denote the representing *G*-space for F_V. Then

$$[X, \Omega^V BF_V]^G = [S^V \wedge X, BF_V]^G = \prod_{(H)} [(S^V \wedge X)^H, L(H)_{V^H}]^{W_G H} =$$

$$\prod_{(H)} [(S^{V^H} \wedge X^H), L(H)_{V^H}]^{W_G H} = \prod_{(H)} [X^H, \Omega^{V^H} L(H)_{V^H}]^{W_G H} =$$

$$\prod_{(H)} [X^H, L(H)]^{W_G H} = [X, J(G,p)]^G$$

proving that $\Omega^V BF_V \simeq J(G,p)$.

QED

Definition 2.8

Let *H* be a subgroup of *G*. We then have the functor $I_H : \mathcal{E}_G{}^H \to \mathcal{E}_{W_G H}$, where an object of $\mathcal{E}_G{}^H$ is considered as an $W_G H$-set.

i_H induces the infinite $W_G H$-loop map

$$i_H = \Omega B(Bi_H) : (Q_G S^0)^H = \Omega B(B\mathcal{E}_G{}^H) \to \Omega B(B\mathcal{E}_{W_G H}) = Q_{W_G H} S^0$$

Another description of i_H is as follows: If *V* is a $\mathbb{R}G$-module, then we have the $W_G H$-map

$$m_V : Map(S^V, S^V)^H \cong Map_H(S^V, S^V) \to Map(S^{V^H}, S^{V^H}),$$

which sends the *H*-map $f : S^V \to S^V$ to the map $f^H : S^{V^H} \to S^{V^H}$. By taking the limit over all $\mathbb{R}G$-modules *V*, we get a $W_G H$-map $m : (Q_G S^0)^H \to Q_{W_G H} S^0$. This map *m* and the map i_H from above are $W_G H$-homotopic maps.

Definition 2.9

The map $e(G, p) : Q_G S^0 \to J(G, p)$ is defined as follows: In case I (the *p*-group case), we let the *H*'th component of

$$e(G, p) \in [Q_G S^0, J(G, p)] = \prod_{(H)} [(Q_G S^0)^H, L(H)]^{W_G H}$$

be the composite

$$(Q_G S^0)^H \xrightarrow{i_H} W_{W_G H} S^0 \xrightarrow{e_{wGH}} K(\mathbb{F}_q, W_G H)$$
$$\xrightarrow{j \times Id} K(\mathbb{F}_q) \times K(\mathbb{F}_q, W_G H) = L(H)$$

when $H \neq G$, and the composite

$$(Q_G S^0)^H \xrightarrow{i_G} Q_1 S^0 \xrightarrow{e_1} K(\mathbb{F}_q; 1) = K(\mathbb{F}_q) = L(G)$$

when $H = G$. In case II we let the *H*'th component of

$$e(G, p) \in [Q_G S^0, J(G, p)] = \prod_{(H)} [(Q_G S^0)^H, K(\mathbb{F}_q)]^{W_G H}$$

be the composite

$$(Q_G S^0)^H \xrightarrow{i_H} W_{W_G H} S^0 \xrightarrow{e_{wGH}} K(\mathbb{F}_q, W_G H) \xrightarrow{j} K(\mathbb{F}_q, 1) = K(\mathbb{F}_q)$$

Here $j : K(\mathbb{F}_q, W_G H) \to K(\mathbb{F}_q, 1) = K(\mathbb{F}_q)$ is induced by the group homomorphism $W_H G \to 1$, and 1 denotes the trivial group.

Definition 2.10

The *G*-space $\operatorname{Cok} J(G, p)$ is defined as the *G*-homotopy fibre of the map

$$e(G, p) : Q_G S^0 \to J(G, p)$$

Proposition 2.11

The map $e(G,p): Q_G S^0 \to J(G,p)$ is an infinite G-loop map, both in case I and in case II.

Proof:

We have that the H'th component of $e(G,p)$ in both cases is an infinite $W_G H$-loop map, as it is composed of infinite $W_G H$-loop maps. We now proceed as in the proof of (2.7).

QED

For later use we calculate the map $e(G,p):(Q_G S^0)^G \to J(G,p)^G$, when G is a cyclic p-group. (In the next section we will describe $\Phi e(G,p): \Phi Q_G S^0 \to \Phi J(G,p)$ in the case Il, where $(|G|,p)=1$)

Recall from [H91] , (4.5), that we have a map $e: Q(BG_+) \to K(\mathbb{F}_q[G])$ defined as follows: Consider the categories $\mathcal{T}_G$ and $\mathcal{GL}(\mathbb{F}_q[G])$ of G-sets of the form $n(G/1)$, $n \geq 0$, and projective $\mathbb{F}_q[G]$-modules, respectively. These categories have classifying spaces $\Omega B(B\mathcal{T}_G) = Q(BG_+)$ and $\Omega B(B\mathcal{GL}(\mathbb{F}_q[G])) = K(\mathbb{F}_q[G])$.

The functor $P:\mathcal{T}_G \to \mathcal{GL}(\mathbb{F}_q[G])$, sending a G-set to its permutation representation: $P(n(G/1)) = \mathbb{F}_q[G]^n$, gives the infinite loop map

$$e = \Omega B(BP): Q(BG_+) = \Omega B(B\mathcal{T}_G) \to \Omega B(B\mathcal{GL}(\mathbb{F}_q[G])) = K(\mathbb{F}_q[G]).$$

Proposition 2.12

Let G be a cyclic p-group; $G = \mathbb{Z}/p^n$. Let $1 = G_0 \subset G_1 \subset ... \subset G_n = G$ be a complete list of the subgroups of G. Then the matrix of the map

$$e(G,p)^G : (Q_G S^0)^G = \prod_{t=0}^{n} Q(B(G/G_t)_+) \to \prod_{s=0}^{n} L(G_s) = J(G,p)^G$$

has the (s,t)'th entry e_{st} given by

$$e_{st} = \begin{cases} p^{s-t}(j \circ e) \times e & n > s \geq t \\ p^{s-t} e & n = s \geq t \\ 0 & s < t \end{cases}$$

The map j is described in (2.9).

Proof:

Let $K \leq H \leq G$ be subgroups. Consider the composite

$$Q(B/G/K)_+) \xrightarrow{a} (Q_G S^0)^G = ((Q_G S^0)^H)^{G/H} \xrightarrow{(i_H)^{G/H}} (Q_{G/H} S^0)^{G/H}$$

where a is the inclusion of the factor, cf. (1.2), and i_H is the map from (2.8). By considering this composite as the realization of the functor, which restricts a G-set of the form $m(G/K)$ to a G/H-set, we see that $m(G/K)$ is mapped to $mp^{s-t}(G/H)$, where $|K| = p^t$ and $|H| = p^s$.

As $e(G,p)$ on the fixed point sets is composition of the map above with $(j \times Id) \circ e$, we get the result.

QED

3. The Sullivan splitting

In this section we only consider case II, i.e. we assume that $(|G|, p) = 1$. *p* is, as usual, an odd prime. We show that we have a *p*-local splitting

$$(SF_G)_{(p)} \simeq (J(G,p)_0)_{(p)} \times (\mathrm{Cok}\, J(G,p)_0)_{(p)},$$

where $J(G,p)_0$ and $\mathrm{Cok}\, J(G,p)_0$ denote the *G*-connected covers of $J(G,p)$ and $\mathrm{Cok}\, J(G,p)$, respectively, cf. [El], p.277.

Proposition 3.1

Let *p* be a prime not dividing the order of the group *G*. Let *X* be a *G*-space, and let *Y* be a *p*-local infinite *G*-loop space. Then the map

$$Fix : ([X,Y]^G)_{(p)} \to ([\Phi X, \Phi Y]_{\mathcal{O}_G})_{(p)}$$

sending the *G*-map $f : X \to Y$ to the $\mathcal{O}_G$-map

$$Fix(f) : G/H \mapsto (f^H : X^H \to Y^H)$$

is a bijection.

Proof:

This is essentially [LMS], (V.6.8) and (V.6.9): If $(|G|, p) = 1$, then

$$[X,Y]^G{}_{(p)} \cong \prod_{(H)} [X^H, Y^H]_{(p)}{}^{INV},$$

where the superscript *INV* indicates that we are considering homotopy classes of 'invariant maps', [LMS] (V.6.5). But such an invariant homotopy class corresponds to a $\mathcal{O}_G$-homotopy class of $\mathcal{O}_G$-maps $\Phi X \to \Phi Y$.

QED

Proposition 3.2

For a subgroup *H* of *G* the fixed point map

$$e(G,p)^H_{(p)} : (Q_G S^0{}_{(p)})^H = \prod_{(K)_H} QS^0{}_{(p)} \to \prod_{(K)_H} K(\mathbb{F}_q)_{(p)} \to J(G,p)^H_{(p)}$$

is given by the product of maps $e_{(p)} : QS^0{}_{(p)} \to K(\mathbb{F}_q)_{(p)}$.

Proof:

Let $K \le H \le G$ be subgroups, and denote $W_G H$ by *W*. The composite

$$Q(BW_H K_+)_{(p)} \xrightarrow{a} (Q_G S^0{}_{(p)})^H \xrightarrow{i_H} Q_W S^0{}_{(p)} \xrightarrow{e_W} K(\mathbb{F}_q, W)_{(p)} \xrightarrow{j} K(\mathbb{F}_q)_{(p)}$$

is zero, if $K < H$, and equal to $e_{(p)} : QS^0{}_{(p)} \to K(\mathbb{F}_q)_{(p)}$, if $K = H$ =H. This is seen by using discrete models: The *H*-set $n(H/K)$ is mapped via $e_W \circ i_H \circ a$ to the $\mathbb{F}_q[W]$-module $\mathbb{F}_q[W/K]^n$. But the categorical analogue of *j* maps a $\mathbb{F}_q[W]$-module *V* to its fixed point module V^W.

QED

We briefly review the non equivariant case, see [M77] p. 112 ff.: We have *p*-local infinite loop maps

(3.3) $\alpha : J_p \to SF$ and $e : SF \to J_p$,

such that $e \circ \alpha : J_p \to J_p$ is a homotopy equivalence. This gives a splitting

(3.4) $SF \simeq J_p \times \operatorname{Cok} J_p$

where $\operatorname{Cok} J_p$ is the homotopy fibre of $e : SF \to J_p$. The work of Quillen in [Q], thm.7, p.585, shows that J_p can be identified with the connected cover of $K(\mathbb{F}_q)$ – algebraic *K*-theory of the finite field $\mathbb{F}_q$. Finally, the map $e : SF \to J_p$ is actually the connected cover of the map $e : Q(BG_+) \to K(\mathbb{F}_q[G])$ of (1.7), with *G* being the trivial group.

Definition 3.5

Define $\alpha(G,p) : J(G,p)_0 \to SF_G$ as follows: The $\mathcal{O}_G$-map

$$\Phi\alpha(G,p) : \Phi J(G,p)_0 \to \Phi SF_G$$

is defined to be

$$\Phi\alpha(G,p)(G/H) = \prod_{(K)_H} \alpha : J(G,p)_0{}^H = \prod_{(K)_H} K(\mathbb{F}_q)_0 \to \prod_{(K)_H} SF = SF_G{}^H$$

By using (3.1), we let $\alpha(G,p) : J(G,p)_0 \to SF_G$ correspond to $\Phi\alpha(G,p) \in [\Phi J(G,p)_0, \Phi SF_G]_{\mathcal{O}_G}$.

Theorem 3.6

$e(G,p)$ and $\alpha(G,p)$ induces a *p*-local splitting

$$SF_G \simeq J(G,p)_0 \times \operatorname{Cok} J(G,p)_0$$

Proof:
This follows immediately from the fact that the composite $e(G,p)\circ\alpha(G,p)$ is a *G*-homotopy equivalence: For each subgroup *H* of *G* the fixed point map $(e(G,p)\circ\alpha(G,p))^H$ is simply $\prod_{(K)_H} e\circ\alpha$, and from (3.4) we conclude that this is a homotopy equivalence.

QED

4. K_G – theory of $\mathrm{Cok}\, J(G,p)$

In this section we show that K_G-theory of $\mathrm{Cok}\, J(G,p)$ vanishes. To this purpose we prove some results linking K_G-theory of a *G*-Space to the *K*-theory of the fixed point sets. These results will be applied later on.

In order to avoid finiteness assumptions, we agree to work with *K*-theory with coeffecients in $\mathbb{Z}/p$ for an odd prime *p*. We remark that the results (4.1) and (4.2) holds for *K*-theory with integral coeffecients, provided that *X* is a finite *G-CW*-complex.

Lemma 4.1

Let *G* be a cyclic group, *X* a *G-CW*-complex with $\bar{K}^*(X^H)=0$ for every subgroup *H* of *G*. Then $\bar{K}_G{}^*(X)=0$.

(Here $\bar{K}_G(-)$ is *reduced* K_G*-theory* – for a finite *G-CW*-complex *X*, $\bar{K}_G(X)$ is generated by differences of *G*-bundles E_F satisfying the following condition: For every $x\in X$, the fibres E_x and F_x are equivalent G_x-modules.)

Proof:

We show this using induction over the subgroups of *G*. Let H_1, H_2 ..., H_n be an ordering of these subgroups, such that if $H_i\subseteq H_j$,then $i\geq j$. Since *G* is cyclic, every H_i is normal in *G*, and we let W_i denote the factor group G/H_i.

Define a filtration $X_1\subseteq X_2\subseteq ...\subseteq X_n$ of *X* by

$$X_i=\bigcup_{j=1}^{i} X^{H_j}$$

Since all subgroups H_j are normal in *G*, each X^{H_j} is a *G*-space, and thus X_i is a *G*-space. We show inductively that $\bar{K}_G{}^*(X_i)=0$.

For $i=1$ we have that $H_1=G$, and thus

$$\bar{K}_G{}^*(X_1)=\bar{K}_G{}^*(X^G)=\bar{K}^*(X^G)\otimes R(G)=0.$$

Assume now that $\bar{K}_G{}^*(X_{i-1})=0$. By considering the long exact sequence

$$\to\bar{K}_G{}^{*-1}(X_{i-1})\to\bar{K}_G{}^*(X_i,X_{i-1})\to\bar{K}_G{}^*(X_i)\to\bar{K}_G{}^*(X_{i-1})\to$$

coming from the pair (X_i,X_{i-1}), we see that $\bar{K}_G{}^*(X_i)\cong\bar{K}_G{}^*(X_i,X_{i-1})$.

Since we have the canonical group homomorphism $G \to W_i$, we get a G-action on the space EW_i. We have that $(EW_i)^H$ is either contactible or the empty set Ø, depending on whether H is contained in H_i or not.

The projection map between the pairs $(EW_i \times X_i, EW_i \times X_{i-1})$ and (X_i, X_{i-1}) is a G-homotopy equivalence, since it is an homotopy equivalence on all fixed point sets:

For every subgroup H_j one of the following three conditions is satisfied:

a) $j < i$, or
b) $j \geq i$ and $H_j \subseteq H_i$, or
c) $j \geq i$ and $H_j \not\subseteq H_i$.

If condition a) is satisfied, then $(X_i)^{H_j} = (X_{i-1})^{H_j}$, and $(X_i / X_{i-1})^{H_j}$ is contractible. Whether $EW_i^{H_j}$ is the zero set or contractible, the set $(EW_i \times X_i / EW_i \times X_{i-1})^{H_j}$ is contractible.

If condition b) is satisfied, then $(EW_i)^{H_j}$ is contractible, and the claim is trivial.

If condition c) is satisfied, then $(EW_i)^{H_j} = \varnothing$, and $(EW_i \times X_i / EW_i \times X_{i-1})^{H_j}$ is contractible. But $(X_i)^{H_j} = (X_{i-1})^{H_j} \cup (X^{H_i})^{H_j}$, and the set $(X^{H_i})^{H_j} = X^{H_k}$ is contained in $(X_{i-1})^{H_j}$; here H_k is the smallest subgroup of G-containing the subgroups H_i and H_j, and since $H_k \supset H_i$, we know that $k < i$, and $X^{H_k} \subseteq X_{i-1}$.

We thus have $\bar{K}_G{}^*(X_i) \cong \bar{K}_G{}^*(X_i, X_{i-1}) \cong \bar{K}_G{}^*(EW_i \times X_i, EW_i \times X_{i-1})$

We note that H acts trivially on the spaces EW_i and X_i / X_{i-1}. [MR84], (1.3.12), stating that if G is an Abelian group, Γ a subgroup of G, then there is an isomorphism

$$K_G(X^\Gamma) \cong K_{G/\Gamma}(X^\Gamma) \otimes R(\Gamma),$$

reduces the problem to showing that $\bar{K}_{W_i}{}^*(EW_i \times X_i, EW_i \times X_{i-1})$ is zero. By using the long exact sequence on the pair $(EW_i \times X_i, EW_i \times X_{i-1})$, we see that it suffices to show that the restriction map $\bar{K}_{W_i}{}^*(EW_i \times X_i) \to \bar{K}_{W_i}{}^*(EW_i \times X_{i-1})$ is an isomorphism.

W_i acts freely on EW_i, and by using [S68c] (2.1) it suffices show that the map $\bar{K}^*(EW_i \times_{W_i} X_i) \to \bar{K}^*(EW_i \times_{W_i} X_{i-1})$ is an isomorphism.

We have the homotopy commutative diagram of fibrations

$$\begin{array}{ccc} X_i & \to & X_{i-1} \\ \downarrow & & \downarrow \\ EW_i \times_{W_i} X_i & \to & EW_i \times_{W_i} X_{i-1} \\ \downarrow & & \downarrow \\ BW_i & = & BW_i \end{array}$$

Using a Mayer-Vietoris argument, we conclude that $\bar{K}^*(X_i) = \bar{K}^*(X_{i-1}) = 0$. The Atiyah-Hirzebruch spectral sequence and the comparision theorem for spectral sequences show that $\bar{K}^*(EW_i \times_{W_i} X_i) \to \bar{K}^*(EW_i \times_{W_i} X_{i-1})$ is an isomorphism.

QED

<u>Lemma 4.2</u> ([MC], cor. C.)

Let *G* be a finite group, *p* an odd prime. Let *X* be a *G*-*CW*-complex, such that $\bar{K}_C{}^*(X) = 0$ for every cyclic subgroup of *G*. Then $\bar{K}_G{}^*(X) = 0$.

<u>Proposition 4.3</u>

Let $\mathrm{Cok}\, J(G,p)$ be the space from (2.10). Then $\bar{K}_G{}^*(\mathrm{Cok}\, J(G,p);\mathbb{Z}/p)$ vanishes.

<u>Proof:</u>

According to (4.l) and (4.2) it suffices to show that $e(G,p)^K : (Q_G S^0)^K \to J(G,p)^K$ is an $K^*(-;\mathbb{Z}/p)$-equivalence, when *G* is a cyclic group, and *K* is a subgroup of *G*.

In case I, the *p*-group case, we use (2.12) and the fact, that

$$(j \times Id) \circ e : Q(B\Gamma_+) \to K(\mathbb{F}_q) \times K(\mathbb{F}_q, \Gamma)$$

is a $K^*(-;\mathbb{Z}/p)$-equivalence, when Γ is a cyclic *p*-group, cf. (1191), (4.18).

In the case where $|G|$ is invertible in $\mathbb{Z}/p$, we use the definition (3.2) and the fact that $e : QS^0 \to K(\mathbb{F}_q)$ is a $K^*(-;\mathbb{Z}/p)$-equivalence, cf. [MM], (5.22).

QED

From (2.11) we know that $\mathrm{Cok}\, J(G,p)$ is an infinite *G*-loop space. We want to show that K_G-theory of the corresponding *G*-spectrum vanishes.

<u>Lemma 4.4</u>

Let *X* be an infinite *G*-loop space with $\bar{K}^*(X^H) = 0$ for every subgroup *H* of *G*. Let *V* be an $\mathbb{R}G$-module. Then the *V*'th delooping X_V of *X* satisfies $\bar{K}^*(X_V{}^H) = 0$ for every subgroup *H*. Especially, $\bar{K}_G{}^*(X) = 0$.

<u>Proof:</u>

This follows from (the dual) of the *K*-theoretical Rothenberg-Steenrod spectral sequence, cf. [Ho], p.5 (the dualization is carried through in [McC], p.242 ff):

$$E^2 = \mathrm{Tor}_{K_*(X)}(\mathbb{Z}_{(p)}, \hat{\mathbb{Z}}_p) \Rightarrow K_*(BX) = E^\infty$$

QED

Corollary 4.5

The K_G-theory with coeffecients in $\mathbb{Z}/p$ of the spectra $\mathrm{Cok}\, J(G,p)$ vanishes.

Corollary 4.6

The map of G-spectra $e(G,p): S_G \to J(G,p)$ of (2.9) is an $K_G(-;\mathbb{Z}/p)$-equivalence.

We here denote the G-sphere spectrum with S_G, and we denote the G-spectra corresponding to the infinite G-loop spaces $J(G,p)$ and $\mathrm{Cok}\, J(G,p)$ by the same symbols, i.e. by $J(G,p)$ and $\mathrm{Cok}\, J(G,p)$

5. Equivariant Bousfield localization

In this section we briefly describe the properties of equivariant Bousfield-localization. The formal development of the theory will not be done here, as the nonequivariant results of [B79a] and [B79b] immediately can be generalized to the equivariant case:

We work in the category $Ho_G{}^S$ of G-CW-spectra as described in [LMS], p.27 ff. This is the homotopy category of spectra indexed over the complete G-universe $\mathcal{U}$ consisting of countably many copies of the regular representation $\mathbb{R}[G]$. Every object in $Ho_G{}^S$ is assumed to be a G-cell spectrum, i.e. it has a decomposition into stable G-cells of the form $\Sigma^n(S_G \wedge (G/H)_+)$, where n ranges over the integers, and H over the subgroups of G.

Definition 5.1

Let A be a G-spectrum. A G-spectrum X is A-acyclic, if $A^*(X)=0$. A map $f: X \to Y$ in $Ho_G{}^S$ is an A-equivalence, if the map $A^*(f): A^*(X) \to A^*(Y)$ is an isomorphism. The G-spectrum B is A-local if, for every A-equivalence $f: X \to Y$, the map $f^*: [Y,B]_* \to [X,B]_*$ is an isomorphism. Finally, the map $f: X \to Y$ is an A-localization of X, if f is an A-equivalence, and Y is A-local.

Analogous to [B79b] (1.1), we have

Theorem 5.2

Every G-spectrum X in $Ho_G{}^S$ has an A-localization, denoted by X_A or $L_A X$. This A-localization is unique up to equivalence.

The most obvious examples of equivariant Bousfield localizations are localization with respect to Eilenberg-MacLane speetra, cf. [LMM9. Especially, if $X = H_G(\underline{\mathbb{Z}}_{(p)},0)$, where $\underline{\mathbb{Z}}_{(p)}$ denotes the constant coeffecient system $G/H \mapsto \mathbb{Z}_{(p)}$, then L_X is the localization at the prime *p* of [MMT]. Similarly, localization with respect to $H_G(\underline{\mathbb{F}}_q,0)$ gives the *p*-adic completion of [M81].

Both these localization functors have the property that they preserve the formation of fixed point sets:

Proposition 5.3

Let *X* be one of the spectra $H_G(\underline{\mathbb{Z}}_{(p)},0)$ or $H_G(\underline{\mathbb{F}}_q,0)$. Let *A* be any *G*-spectrum, and let M be a subgroup of *G*. Then

$$L_{X^M}(A^M) \simeq (L_X A)^M$$

Proof:

This is thm. 10 of [MMT] and thm, 14 of [M81].

QED

For general spectra *X*, the statement of (5.3) does probably not hold. We therefore introduce a variant of K_G-theory, which in the equivariant case is more manageable:

Definition 5.4

Let $\mathcal{K}_G$ be the *G*-spectrum defined by $\mathcal{K}_G = \bigvee_{(H)} K \wedge (G/H_+)$, where the wedge is over all conjugacy classes of subgroups of *G*, K is the spectrum representing ordinary *K*-theory, and G/H_+ is the usual *G*-space.

Proposition 5.5

1) *X* is $\mathcal{K}_G$-acyclic if and only if $K_*(X^H) = 0$ for every subgroup *H* of *G*.
2) Every $\mathcal{K}_G$-local *G*-spectrum is K_G-local.

Proof:

1) is obvious. In order to prove 2), let *X* be a $\mathcal{K}_G$-acyclic *G*-spectrum. This implies that for every subgroup *H* of *G* we have that $[K \wedge (G/H_+), X]^G = [K, X^H] = 0$. Thus X^H is *K*-local, and $K^*(X^H) = 0$. (3.2) now implies that $K_G(X) = 0$.

QED

It follows immediately from the definition (5.4) that

Proposition 5.6

Let *X* be a *G*-spectrum, *M* a subgroup of *G*. Then

$$(L_{\mathcal{K}_G}(X))^M \simeq L_K(X^M).$$

6. The equivariant *K*-localization of the *G*-sphere spectrum

In this section we calculate the $\mathcal{K}_G$ -localization of the equivariant sphere-spectrum S_G . Again, all spectra are assumed to be *p*-local, and we work with the two cases:

I: *G* is a *p*-group, and
Il: *G* is a finite group with $(p,|G|)=1$.

Out starting point is (4.6), which we immediately generalize to *p*-adic coeffecients:

Proposition 6.1

The map of *G*-spectra $e(G,p):S_G \to J(G,p)$ of (2.9) is an $\mathcal{K}_G(-;\hat{\mathbb{Z}}_p)$-equivalence.

Proof:

By using the Bockstein sequences coming from the coeffecient sequences

$$0 \to \mathbb{Z}/p^{n-1} \to \mathbb{Z}/p^n \to \mathbb{Z}/p \to 0,$$

we see inductively that $e(G,p)$ is a $\mathcal{K}_G(-;\mathbb{Z}/p^n)$ equivalence, i.e. for every subgroup H of *G* the map $e(G,p)^H : S_G{}^H \to J(G,p)^H$ is a $K(-;\mathbb{Z}/p^n)$-equivalence.

Let *X* be a spectrum, and let S^0A be the Moore spectrum for the Abelian group *A*, see [A74], p.200. We have that $S^0\hat{\mathbb{Z}}_p \cong \underleftarrow{\lim} S^0\mathbb{Z}/p^n$, and thus

$$K^*(X;\hat{\mathbb{Z}}_p) \cong [X;K\wedge S^0\hat{\mathbb{Z}}_p]_* \cong [X\wedge DS^0\hat{\mathbb{Z}}_p;K]_* \cong [X\wedge \underrightarrow{\lim} DS^0\mathbb{Z}/p^n;K]_*,$$

where DZ is the Spanier-Whitehead dual of the spectrum *Z*.

The Milnor sequence applied on the space $X\wedge \underrightarrow{\lim} DS^0\mathbb{Z}/p^n$ now gives the natural exact sequence

$$0 \to \underleftarrow{\lim}^1 K^{*-1}(X;\mathbb{Z}/p^n) \to K^*(X;\hat{\mathbb{Z}}_p) \to \underleftarrow{\lim} K^*(X;\mathbb{Z}/p^n) \to 0$$

By applying this sequence on the map $e(G,p)^H : S_G{}^H \to J(G,p)^H$, we obtain the diagram

$$\begin{array}{ccccccc}
0 \to & \underleftarrow{\lim}^1 K^{*-1}(S_G{}^H;\mathbb{Z}/p^n) & \to & K^*(S_G{}^H;\hat{\mathbb{Z}}_p) & \to & \underleftarrow{\lim} K^*(S_G{}^H;\mathbb{Z}/p^n) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \underleftarrow{\lim}^1 K^{*-1}(J(G,p)^H;\mathbb{Z}/p^n) & \to & K^*(J(G,p)^H;\hat{\mathbb{Z}}_p) & \to & \underleftarrow{\lim} K^*(J(G,p)^H;\mathbb{Z}/p^n) & \to 0
\end{array}$$

As the first and last vertical arrows are isomorphisms, we conclude that the middle arrow is an isomorphism, too, and the result now follows from (5.5).

QED

This almost calculates the $\mathcal{K}_G$-localization of S_G. But there are two difficulties: $J(G,p)$ is not $\mathcal{K}_G$-local *G*-spectrum, and $e(G,p)$ is not a $\mathcal{K}_G(-;\mathbb{Q})$-equivalence. We remedy this:

Recall from [FHM], (0.5), that we have a cofiber sequence

(6.2) $$K(\mathbb{F}_q,G) \to K_G < 0,\infty > \xrightarrow{\psi^q - 1} K_G < 2,\infty >$$

where $K_G < n,\infty >$ is the *n*-connected cover of the *G*-spectrum K_G. Define $\mathcal{K}(\mathbb{F}_q,G)$ as the homotopy fibre of $\psi^q - 1 : K_G \to K_G$.

We remark that $\mathcal{K}(\mathbb{F}_q,G)$ is a K_G-local *G*-spectrum, as K_G is K_G-local. Furthermore, $\mathcal{K}(\mathbb{F}_q,G)$ is $\mathcal{K}_G$-local: For a subgroup *H* of *G* we have the cofiber sequence

$$\mathcal{K}(\mathbb{F}_q,G)^H \to {K_G}^H \xrightarrow{\psi^q - 1} {K_G}^H$$

It follows from [FHM], (3.1), that ${K_G}^H$ is equivalent to $\bigvee_{Irr(H)} K$, where the wedge is over all inequivalent, irreducible $\mathbb{C}H$-modules. (*K* is the (non-equivariant) spectrum representing ordinary complex *K*-theory). As ${K_G}^H$ is *K*-local, $\mathcal{K}(\mathbb{F}_q,G)^H$ is *K*-local, and it follows that $\mathcal{K}(\mathbb{F}_q,G)$ is $\mathcal{K}_G$-local.

Proposition 6.3

The map $a : K(\mathbb{F}_q,G) \to \mathcal{K}(\mathbb{F}_q,G)$ coming from the diagram

$$\begin{array}{ccccc}
K(\mathbb{F}_q,G) & \to & K_G < 0,\infty > & \xrightarrow{\psi^q - 1} & K_G < 2,\infty > \\
{\scriptstyle a}\downarrow & & \downarrow & & \downarrow \\
\mathcal{K}(\mathbb{F}_q,G) & \to & K_G & \xrightarrow{\psi^q - 1} &
\end{array}$$

is a $\mathcal{K}_G(-;\hat{\mathbb{Z}}_p)$-equivalence.

Proof:

We show that for every subgroup *H* of *G* the map $a^H : K(\mathbb{F}_q,G)^H \to \mathcal{K}(\mathbb{F}_q,G)^H$ is a $K_*(-;\mathbb{Z}/p)$-equivalence.

Let *F* denote the fibre of a^H. It then suffices to show that $K_*(F;\mathbb{Z}/p)$ vanishes. Let F_m denote the *m*-connected cover of *F*. We have the sequence of maps

$$0 = F_1 \to F_0 \to F_{-1} \to F_{-2} \to \ldots$$

and $F = \underrightarrow{\lim} F_{-n}$. As $K_*(F;\mathbb{Z}/p) = \underrightarrow{\lim} K_*(F_{-n};\mathbb{Z}/p)$, it suffices to show that $K_*(F_{-n};\mathbb{Z}/p) = 0$.

This is done inductively: For $n < 0$, F_{-n} is the zero spectrum. We obtain F_{-n} from F_{-n+1} via the cofiber sequence

$$F_{-n+1} \to F_{-n} \to H(\pi_{-n}(F);-n)$$

where $H(\pi_{-n}(F);-n)$ is the Eilenberg-MacLane spectrum with the sole non-zero homotopy group

$$\pi_{-n}(H(\pi_{-n}(F);-n)) = \pi_{-n}(F)$$

Now, $K_*(H(\pi_{-n}(F);-n);\mathbb{Z}/p) = K_*(H(\pi_{-n}(F) \otimes_{\mathbb{Z}} \mathbb{Z}/p;-n) = 0$, as it follows from [AH], thm. 1, and inductively we see that $K_*(F_{-n};\mathbb{Z}/p) = 0$.

QED

Definition 6.4

Assume we are in case 1, i.e. that *G* is a *p*-group. Let $\mathcal{J}(G,p)$ be the *G*-spectrum representing the functor *F* from $Ho_G{}^S$ to Abelian groups given by

$$\mathcal{F}(X) = \prod_{(H)} [X^H, \mathcal{L}(H)]_*^{W_G H}$$

Here $\mathcal{L}(H)$ is the $W_G H$-spectrum $\mathcal{K}(\mathbb{F}_q) \times \mathcal{K}(\mathbb{F}_q, W_G H)$ in case $G \neq H$, and $\mathcal{L}(G)$ is $\mathcal{K}(\mathbb{F}_q)$.

Let $b : J(G,p) \to \mathcal{J}(G,p)$ be the map representing the natural transformation

$$F(X) = \prod_{(H)} [X^H, L(H)]^{W_G H} \xrightarrow{\Pi \bar{a}_*} \prod_{(H)} [X^H, \mathcal{L}(H)]^{W_G H} = \mathcal{F}(X)$$

where $\bar{a}$ is the map $L(H) = K(\mathbb{F}_q) \times K(\mathbb{F}_q, W_G H) \xrightarrow{a \times a} \mathcal{K}(\mathbb{F}_q) \times \mathcal{K}(\mathbb{F}_q, W_G H) = \mathcal{L}(H)$ when $G \neq H$ i#11, and $\bar{a} : L(G) = K(\mathbb{F}_q) \to \mathcal{K}(\mathbb{F}_q) = \mathcal{L}(G)$ is simply *a* itself.

In case II, where $(|G|, p) = 1$, we let $\mathcal{J}(G,p)$ be the *G*-spectrum representing the functor $\mathcal{F}$ from $Ho_G{}^S$ to Abelian groups given by

$$\mathcal{F}(X) = \prod_{(H)} [X^H, \mathcal{K}(H)]_*^{W_G H}$$

and where $\mathcal{K}(\mathbb{F}_q) = K(\mathbb{F}_q, 1)$. In this case, the map $b : J(G,p) \to \mathcal{J}(G,p)$ is representing the natural transformation

$$F(X) = \prod_{(H)} [X^H, K(H)]^{W_G H} \xrightarrow{\Pi \bar{a}_*} \prod_{(H)} [X^H, \mathcal{K}(H)]^{W_G H} = \mathcal{F}(X)$$

Proposition 6.5

1) $b : J(G,p) \to \mathcal{J}(G,p)$ is a $\mathcal{K}_G(-;\hat{\mathbb{Z}}_p)$-equivalence, and

2) $\mathcal{J}(G,p)$ is $\mathcal{K}_G$-local.

Proof:

l) follows immediately from (6.3) by calculating the action of *b* at the fixed point spectra – cf. (2.4).

In order to show 2), let *X* be a $\mathcal{K}_G$-acyclic *G*-spectrum. Then we have for every subgroup *H* of *G* and every subgroup *V* of $W_G H$ that $(X^H)^V$ is *K*-acyclic. It follows from (4.1) that X^H is $K_{W_G H}$-acyclic, and $[X^H, \mathcal{K}(\mathbb{F}_q) \times \mathcal{K}(\mathbb{F}_q, W_G H)]_*^{W_G H} = 0$, as both $\mathcal{K}(\mathbb{F}_q)$ and $\mathcal{K}(\mathbb{F}_q, W_G H)$ are $K_{W_G H}$-local. Thus $[X, \mathcal{J}(G,p)]_*^G = 0$.

The proof in case II is similar.

QED

We now study the rational type of S_G. We recall from (1.2) that the sole non-zero rational homotopy group of S_G is in dimension 0, and that $\underline{\pi}_0(S_G;\mathbb{Q}) = \underline{A} \otimes \mathbb{Q}$, where the $\mathcal{O}_G$-group $\underline{A} \otimes \mathbb{Q}$ is given by $\underline{A} \otimes \mathbb{Q}(G/H) = A(H) \otimes \mathbb{Q}$. $A(H)$ here denotes the Burnside ring of the finite group *H*.

In the *p*-group case (case I) we have that the *G*-spectrum $\mathcal{K}(\mathbb{F}_q, G)$ has two non-zero rational homotopy groups, as it follows from (6.2). Both $\underline{\pi}_0(\mathcal{J}(\mathbb{F}_q, G);\mathbb{Q})$ and $\underline{\pi}_{-1}(\mathcal{J}(\mathbb{F}_q, G);\mathbb{Q})$ equal $(\underline{A} \oplus \underline{R}'_{\mathbb{F}_q}) \otimes \mathbb{Q}$, where the $\mathcal{O}_G$-group $(\underline{A} \oplus \underline{R}'_{\mathbb{F}_q}) \otimes \mathbb{Q}$ is given by

$$(\underline{A} \oplus \underline{R}'_{\mathbb{F}_q})(G/H) = \begin{cases} A(H) \oplus R_{\mathbb{F}_q}(H) & H \neq G \\ A(G) & H = G \end{cases}$$

$R_{\mathbb{F}_q}(H)$ is the Grothendieck group of $\mathbb{F}_q[H]$-modules. It follows that for a subgroup *H* of *G* the spectrum $(\mathcal{J}(G,p))^H = \prod_{(K)_H} \mathcal{L}(H)^{W_H K}$ only has non-zero rational homotopy in dimensions –1 and 0.

Similarly, in case II $\mathcal{J}(G,p)$ has non-zero rational homotopy only in dimensions –1 and 0, and both are given by $\underline{A} \otimes \mathbb{Q}$.

Definition 6.6

Let $\mathcal{J}'(G,p)$ be the homotopy fibre of the map

$$I : \mathcal{J}(G,p) \to H_G(\underline{\pi}_{-1}(\mathcal{J}(G,p));\mathbb{Q}); -1),$$

which induces the identity map at $\underline{\pi}_{-1}(\mathcal{J}(G,p);\mathbb{Q}) = \underline{H}_{-1}(\mathcal{J}(G,p);\mathbb{Q})$. (The Hurewicz map $H : \underline{\pi}_{-1}(\mathcal{J}(G,p);\mathbb{Q}) \to \underline{H}_{-1}(\mathcal{J}(G,p);\mathbb{Q})$ is an isomorphism, as it follows from [Sp], (9.6.15) – Hurewicz' theorem modulo the Serre class consisting of finite, Abelian groups.)

Let $e'(G,p)$ denote the lift of $b \circ e(G,p) : S_G \to \mathcal{J}(G,p)$ to $\mathcal{J}'(G,p)$ – $I \circ b \circ e(G,p) = 0$ as $\pi_{-1}(S_G) = 0$ and thus $b \circ e(G,p)$ lifts.

Proposition 6.7

$\mathcal{J}'(G,p)$ is a $\mathcal{K}_G$-local *G*-spectrum. $e'(G,p): S_G \to \mathcal{J}'(G,p)$ is a $\mathcal{K}_G(-;\hat{\mathbb{Z}}_p)$-equivalence.

Proof:

$H_G(\underline{\pi}_{-1}(\mathcal{J}(G,p);\mathbb{Q});-1)$ is a $\mathcal{K}_G$-local spectrum, as every rational spectrum is *K*-local, see [Mi], p.207. From (6.5.2) it now follows that $\mathcal{J}'(G,p)$ is $\mathcal{K}_G$-local.

Furthermore, as $H_G(\underline{\pi}_{-1}(\mathcal{J}(G,p);\mathbb{Q});-1)$ vanishes *p*-adically, we conclude that the map $\mathcal{J}'(G,p) \to \mathcal{J}(G,p)$, and hence the lifted map $e'(G,p): S_G \to \mathcal{J}'(G,p)$ are $\mathcal{K}_G(-;\hat{\mathbb{Z}}_p)$-equivalences.

QED

Definition 6.8

Let *G* be a *p*-group. Define the $\mathcal{O}_G$-map $\underline{P}: \underline{\pi}_0(\mathcal{J}'(G,p)) \to \underline{R}_{\mathbb{F}_q}(G) \otimes \mathbb{Q}$ as follows:

For a proper subgroup *H* of *G*, $\underline{P}(G/H)$ is the composite

$$\pi_0(\mathcal{J}'(G,p)) = A(H) \oplus R_{\mathbb{F}_q}(H) \xrightarrow{\pi} R_{\mathbb{F}_q}(H) \xrightarrow{r} R_{\mathbb{F}_q}(H) \otimes \mathbb{Q},$$

where π is the projection onto the second factor, while *r* is the 'rationalization map'.
For $H = G$ we let $\underline{P}(G/G)$ be the zero map.

Definition 6.9

Assume *G* is a *p*-group. The $\mathcal{O}_G$-map $\underline{P}(G)$ of (6.8) corresponds to a *G*-map

$$\mathcal{P}: \mathcal{J}'(G,p) \to H_G(\underline{R}_{\mathbb{F}_q}(G) \otimes \mathbb{Q}; 0)$$

Define the *G*-spectrum $\bar{\mathcal{J}}(G,p)$ as the homotopy fibre of $\mathcal{P}$.

In case II, where $(|G|, p) = 1$, we define $\bar{\mathcal{J}}(G,p)$ to be the *G*-spectrum $\mathcal{J}'(G,p)$ of (6.6).

Theorem 6.10

The $\mathcal{K}_G(-;\mathbb{Z}_{(p)})$-localization of S_G is $\bar{\mathcal{J}}(G,p)$.

Proof:

It follows from the definition of $\bar{\mathcal{J}}(G,p)$ that $e'(G,p)$ factors through $\bar{\mathcal{J}}(G,p)$, and that this factored map is a $\mathcal{K}_G(-;\mathbb{Z}/p)$-equivalence. A direct inspection reveals that this factored map is a $\mathcal{K}_G(-;\mathbb{Q})$-equivalence.

QED

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