The *K*-localizations of Some Classifying Spaces

Kenneth Hansen

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The two other parts are

Oriented, Equivariant *K*-theory and the Sullivan Splitting
The Equivariant *K*-localization of the *G*-Sphere Spectrum

The K-localizations of Some Classifying Spaces
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In this paper the K-theoritical localizations of the suspension spectra for the spaces $\mathbb{C}P^\infty$ and $B\mathbb{Z}/p^n$, where p is an odd prime, are calculated. The result in the case af $\mathbb{C}P^\infty$ is already known; it is to be found in [R], (9.2). The methods used here are completely new, although.

This paper is split into four parts; the first two which contains specific calculations of K-homology groups. In section 1 K-theory of $\mathbb{C}P^\infty$ and $B\mathbb{Z}/p^n$ is described, while section 2 calculates K-homology of the spectra K and $K(\mathbb{F}_l)$. These last calculations rely heavily on the result of Adams, [A74], p. 100, describing the behaviour of the Bott-map in K-theory.

In section 3 the K-localizations of $\Sigma^\infty\mathbb{C}P^\infty$ and of the corresponding infinite loop space $Q(\mathbb{C}P^\infty)$ are calculated, and finally in section 4 we relate the K-localization of $\Sigma^\infty B\mathbb{Z}/p^n$ to algebraic K-theory of the group ring $\mathbb{F}_l[\mathbb{Z}/p^n]$.

Throughout the paper we work within the category of spectra as described in [A74]. Especially, we use the following notation:

If X is a spectrum and n an integer, we denote the n-connected cover of X by $X < n,\infty >$.

K is the periodic spectrum representing complex K-theory, and we define K-homology of the spectrum X to be $K_*(X) = \pi_*(K \wedge X)$.

If X is a topological space, then we denote by $\Sigma^\infty X$ the suspension spectrum of X. $\Sigma^\infty X$ is defined to have the n-th space $(\Sigma^\infty X)_n = \Sigma^n X$ for $n \geq 0$, and if n is negative, then $(\Sigma^\infty X)_n$ is the trivial space.

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1. K-theory of $\mathbb{C}P^\infty$ and $B\mathbb{Z}/p^n$

In this section we study the K-theory of the spaces $\mathbb{C}P^\infty = BS^1$ and $B\mathbb{Z}/p^n$, where p is a fixed, odd prime.

Proposition 1.1 ([A62], (7.2))

Let $n > 0$ be an integer. Then

$$K^0(\mathbb{C}P^n) \cong \mathbb{Z}[\xi]/(\xi^{n+1}) \quad \text{and} \quad K^1(\mathbb{C}P^n) = 0,$$

where $\xi = H - 1$ is the reduced Hopf bundle.

By applying the universal coefficient sequence

(1.2) $$0 \to \mathrm{Ext}_{\mathbb{Z}}(K^{*-1}(X), \mathbb{Z}/p) \to K_*(X;\mathbb{Z}/p) \to \mathrm{Hom}_{\mathbb{Z}}(K^*(X), \mathbb{Z}/p) \to 0$$

of [Y], p.312 and 320, we obtain

Proposition 1.3

(1) $K_0(\mathbb{C}P^n;\mathbb{Z}/p)$ is a free $\mathbb{Z}/p$-module generated by $\beta_0,\beta_1,...,\beta_n$, where β_i is the dual of ξ^i under the isomorphism $K_0(\mathbb{C}P^n;\mathbb{Z}/p)\cong \mathrm{Hom}_{\mathbb{Z}}(K^0(\mathbb{C}P^n),\mathbb{Z}/p)$, i.e.

(1.4) $\quad <\xi^i,\beta_j> = \delta_{ij}$

(2) $K_1(\mathbb{C}P^n;\mathbb{Z}/p)=0$

By taking the limit we get

Corollary 1.5

(1) $K_0(\mathbb{C}P^\infty;\mathbb{Z}/p)$ is a free $\mathbb{Z}/p$-module with the countable basis $\{\beta_0,\beta_1,...\}$.

(2) $K_1(\mathbb{C}P^\infty;\mathbb{Z}/p)=0$.

Let $m\in\mathbb{N}$. Consider the map $\bar{\mu}_m : S^1\to S^1 : \exp(2\pi ix)\mapsto\exp(2\pi imx)$. Define $\mu_m:\mathbb{C}P^\infty\to\mathbb{C}P^\infty$ as $B\bar{\mu}_m$, where we recall that $\mathbb{C}P^\infty = BS^1$.

Proposition 1.6

μ_m restrists to a map $\mathbb{C}P^n\to\mathbb{C}P^n$. The effect of μ_{p^s} in K-homology with coefficients in $\mathbb{Z}/p$ is as follows: $(\mu_{p^s})_*(\beta_i)=\beta_{i/p^s}$ if p^s divides *i*, and is zero otherwise.

Proof:

The composite $\mathbb{C}P^n\to\mathbb{C}P^\infty\xrightarrow{\mu_m}\mathbb{C}P^\infty$ homotopic to a cellular map. As the 2*n*-skeleton of $\mathbb{C}P^\infty$ is $\mathbb{C}P^n$, the image $\mu_m(\mathbb{C}P^n)$ is contained in $\mathbb{C}P^n$, giving the map $\mu_m:\mathbb{C}P^n\to\mathbb{C}P^n$.

Letting H be the Hopf-bundle, we see that $\mu_m{}^*(H)=H^m$. By using the binomial theorem, we get

$$(\mu_{p^s})^*(\xi)=(\mu_{p^s})^*(H-1)=H^{p^s}-1\equiv(H-1)^{p^s}=\xi^{p^s} \pmod p$$

The duality between the ξ^i's and the β_j's gives

$$<\xi^r,(\mu_{p^s})_*\beta_m> = <(\mu_{p^s})^*\xi^r,\beta_m> = <\xi^{rp^s},\beta_m>$$

which is non-zero if and only if $m=rp^s$.

QED

We now turn to the case of $B\mathbb{Z}/p^n$. For the sake of clarity we let G denote $\mathbb{Z}/p^n$, and we let g be the order of G, $g=p^n$.

Recall that we have the *G*-action on S^{2n+1} given as follows: S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1}$. The element $a+g\mathbb{Z}$ of *G* acts on $(z_0, z_1, ..., z_n) \in \mathbb{C}^{n+1}$ by

$$(a+g\mathbb{Z})(z_0, z_1, ..., z_n) = (\eta^a z_0, \eta^a z_1, ..., \eta^a z_n)$$

where $\eta = \exp(2\pi i / g)$. This *G*-action restrists to S^{2n+1}, and the corresponding orbit space is the lens space denoted by $L^n(g)$.

The inclusions $S^{2n+1} \to S^{2n+3}$ gives inclusions $L^n(g) \to L^{n+1}(g)$, and it is readily seen that the space $\lim_{\to} L^n(g)$ is homotopy equivalent to BG.

Furthermore, the standard map $\chi : G \to S^1 : a+g\mathbb{Z} \mapsto \eta^a$ gives rise to maps $B\chi : L^n(g) \to \mathbb{C}P^n$ and $B\chi : BG \to \mathbb{C}P^\infty$.

Let β_i be the dual to $\xi^i = (B\xi)^*(H-1)^i$; i.e. we have the relation

(1.7) $\quad < \xi^i, \beta_j > = \delta_{ij}$.

<u>Proposition 1.8</u>

Let $< \beta_0, \beta_1, ... \beta_{g-1} >$ denote the $\mathbb{Z}/p$-module freely generated by $\beta_0, \beta_1, ..., \beta_{g-1}$. Then

(i) $\quad K_0(BG; \mathbb{Z}/p) = < \beta_0, \beta_1, ..., \beta_{g-1} >$ and

(ii) $\quad K_1(BG; \mathbb{Z}/p) = 0$.

<u>Proof:</u>

This proof is, as that of (1.3), essentially an application of the universal coefficient sequence, (1.2): $K^0(L^n(g))$ is shown in [Ch], thm. 3, to be

$$K^0(L^n(g)) \cong \mathbb{Z}[\xi]/((1+\xi)^g - 1, \xi^{n+1})$$

For $n > g$ we have that $K^0(L^n(g))$ is a free $\mathbb{Z}$-module on the generators $\xi^0, \xi^1, ..., \xi^{g-1}$. By using (1.2) and by taking the limit, we obtain $K_0(BG; \mathbb{Z}/p)$.

An argument using the Atiyah-Hirzebruch spectral sequence shows that $K^1(L^n(g)) \cong \mathbb{Z}$, and that this $\mathbb{Z}$ originates in the top cohomology $H^{2n+1}(L^n(g)) \cong \mathbb{Z}$, which is the only non-zero odd-dimensional cohomology of $L^n(g)$. But the restriction map $K^1(L^{n+1}(g)) \to K^1(L^n(g))$ is zero, and we see that $K_1(BG; \mathbb{Z}/p)$ vanishes.

QED

2. *K*-theory of topological and algebraic K-theory

In this section we continue our calculations. We calculate the K-homology with coefficients in $\mathbb{Z}/p$ of the spectra *K* and $K(\mathbb{F}_{l_i})$, where *K* is the (periodic) spectrum representing complex *K*-theory, and where the spectrum $K(\mathbb{F}_{l_i})$ is algebraic *K* theory of the finite field with l_i elements, $\mathbb{F}_{l_i}$; l_i is assumed to be of the form $l_i = l^{p^i - p^{i-1}}$ for $i > 0$

and $l_0 = l$. l is here an odd prime, such that $l + p^2\mathbb{Z}$ generates the unit group $(\mathbb{Z}/p^2)^\times$ – such primes exist according to a theorem of Dirichlet, cf. [Ap], (7.9).

This last calculation is to be used in §4 – we want to calculate $K_*(K(\mathbb{F}_l[G];\mathbb{Z}/p)$, where G is a cyclic p-group; $G = \mathbb{Z}/p^n$, and we have the splitting of (4.1):

$$K(\mathbb{F}_l[G]) = \prod_{i=0}^{n} K(\mathbb{F}_{l_i})$$

We recall from [A74], p.204, that the spectrum K has the spaces $K_{2n} = BU$ and $K_{2n+1} = U$. The map $B:\Sigma^2 K_{2n} = \Sigma^2 BU \to BU = K_{2n+2}$ is the adjoint of the Bott isomorphism $BU \times \mathbb{Z} \to \Omega^2 BU$.

Denote by β_i also the image of $\beta_i \in K_0(BU(1);\mathbb{Z}/p) = K_0(\mathbb{C}P^\infty;\mathbb{Z}/p)$ under the map $i_H : BU(1) \to BU$ given by the Hopf-bundle. Then we have from [A74], p.47:

Proposition 2.1

(i) $K_0(BU;\mathbb{Z}/p) \cong \mathbb{Z}/p[\beta_1, \beta_2, ...]$

(ii) $K_1(BU;\mathbb{Z}/p) = 0$.

Theorem 2.2

The map $i_* : K_0(BU;\mathbb{Z}/p) \to K_0(BU;\mathbb{Z}/p)$ is surjective. The kernel of i_* is additively generated by all elements decomposable in the β_i's and by the family $\{\gamma_n\}_{n>0}$, where $\gamma_n = (-1)^n \sum_{i=np}^{np+p-1} (-1)^i \beta_i$.

Proof:

Write X for the suspension spectrum of the space BU. The periodicity map B induces a spectrum map $B: X \to \Sigma^{-2} X$, and K is the direct limit spectrum of the system

$$X \xrightarrow{B} \Sigma^{-2} X \xrightarrow{B} \Sigma^{-4} X \xrightarrow{B} ...$$

Thus, $K_0(K;\mathbb{Z}/p)$ is the direct limit of

$$K_0(BU;\mathbb{Z}/p) \xrightarrow{B_*} K_2(BU;\mathbb{Z}/p) \xrightarrow{B_*} ...$$

The map B_* is described in [A74], p. 100: B_* annihilates elements decomposable in β_i's, and

(2.3) $\quad B_*(\beta_i) = u(j\beta_j + (j+1)\beta_{j+1}) + \text{decomposables}, \; j > 0$

where $u = \pi_2(K)$ is the generator (Bott element), cf. [A74], p.38.

Clearly i_* maps all decomposables to zero, and so we need only study B_* on the subspace A of $K_0(BU;\mathbb{Z}/p)$ additively generated by the β_i's.

Split A into submodules A_n, $n = 0,1,2,\ldots$, where A_0 is additively generated by $\beta_1, \beta_2, ... \beta_{p-1}$, and where A_n is additively generated by $\beta_{np}, \beta_{np+1}, ... \beta_{np+p-1}$ for $n > 0$.

Then $A = \bigoplus_{n=0}^{\infty} A_n$, and $B_*(A_n) \subseteq u \cdot A_n + D$, where D is the submodule of $K_0(BU;\mathbb{Z}/p)$ additively generated by all the decomposable elements.

Let $\overline{B}_n$ denote the composite map

$$A_n \xrightarrow{B_*|A_n} K_0(BU;\mathbb{Z}/p) \xrightarrow{\pi_n} A_n$$

where $\pi_n : K_0(BU;\mathbb{Z}/p) \to A_n$ is the projection map. Notice that the eigenvalues of $\overline{B}_n$ are $0, u, 2u, ..., (p-1)u$, for $n > 0$, and $u, 2u, ..., (p-1)u$ for $n = 0$.

The eigenvector corresponding to the eigenvalue 0 for $\overline{B}_n$, where $n > 0$, is easily seen to be

$$\gamma_n = (-1)^n \sum_{i=np}^{np+p-1} (-1)^i \beta_i$$

It is also possible to find eigenvectors corresponding to the other eigenvalues: Let $v = \sum_{i=np}^{np+p-1} a_i\beta_i$ be a vector in A_n. Then $B_*(v) = 0 \cdot \beta_{np} + \sum_{i=np+1}^{np+p-1} i(a_i + a_{i-1})\beta_i$. If v is an eigenvector with eigenvalue ux, $x \neq 0$, then we have the equations:

$$0 = xa_{np} \qquad a_{np} + a_{np+1} = xa_{np+1} \qquad 2(a_{np+1} + a_{np+2}) = xa_{np+2} \qquad \text{etc}$$

These equations can be solved inductively. We have that $a_i = 0$ for $i = 0,1,..., x-1$, $a_x = 1$, and for $i > x$ we have the recurrence relation

$$a_i = ia_{i-1} \cdot (x-i)^{-1}$$

Now, by taking the limit over the B_*'s, we get the result.

QED

For later use we introduce the following

Definition 2.4

Define for integers $n > 0$ and $s \geq 0$ the elements $\gamma_n^{(s)}$ of $K_0(\mathbb{C}P^\infty)$ as follows:

$$\gamma_n^{(0)} = \beta_n$$

$$\gamma_n^{(1)} = \gamma_n \qquad \text{and}$$

$$\gamma_n^{(s)} = (-1)^n \sum_{j=pn}^{pn+p-1} (-1)^j \gamma_j^{(s-1)}$$

Proposition 2.5

(1) $(\mu_p)_*(\gamma_n^{(s)}) = \gamma_n^{(s-1)}$

(2) $(\mu_{p^s})_*(\gamma_n^{(s)}) = \beta_n$

(3) $(\mu_{p^t})_*(\gamma_n^{(s)}) = 0$ for $n \geq 1$, $t \geq n+s+1$.

Proof:

The first equation is shown by induction in s; the main point is that $\mu_{p*}(\gamma_n)=\beta_n$. The second equation follows immediately from the first.

In order to show (3), we observe that the β_i-term in $\gamma_n^{(s)}$ having the most p-divisible index i is β_{np^s}. This term survives at most $s+1+\log_p(n)\leq s+n+1$ attacks by μ_{p*}.

QED

Let q be a prime power. Then there is a cofiber sequence of spectra

(2.6) $$K(\mathbb{F}_q)\xrightarrow{\nu}K<0,\infty>\xrightarrow{\psi^q-1}K<2,\infty>$$

The map $\nu:K(\mathbb{F}_q)\to K<0,\infty>$ is a 'Brauer lift' map as described in e.g. [FP], 166 ff.

Proposition 2.7

Let $l_i=l^{p^i-p^{i-1}}$ for $i>0$ and $l_0=l$. Then, with the notation from (2.2),

(1) $\nu_*:K_0(K(\mathbb{F}_{l_i});\mathbb{Z}/p)\to K_0(K;\mathbb{Z}/p)$ is a monomorphism, whose image is generated by the set $\{i_*(\beta_1),\ldots,i_*(\beta_{p^i-1})\}$, and

(2) $K_1(K(\mathbb{F}_{l_i});\mathbb{Z}/p)=0$.

Proof:

Write, for the sake of simplicity, q instead of l_i. We start by calculating the action of the map $\psi^q-1:\Sigma^\infty BU\to\Sigma^\infty BU$ in K-homology, where $\Sigma^\infty BU$ is the suspension spectrum of the space BU. As the map i_* of (2.2) annihilates decomposable elements, it suffices to calculate ψ^q-1 on the β_i's.

Write $(\psi^q-1)\beta_n=\sum_{j=1}^{\infty}a_{nj}\beta_j$. Then we have

$$a_{nj}=<(\psi^q-1)\beta_n,\xi^j>=<\beta_n,(\psi^q-1)\xi^j>=<\beta_n,g_j(\xi)>=$$
the n'th coefficient in $g_j(\xi)$

where $g_j(\xi)$ is the polynomial given by

$$g_j(\xi)=(\psi^q-1)(\xi^j)=(\psi^q-1)(H-1)^j=(H^q-1)^j-(H-1)^j=((\xi+1)^q-1)^j-\xi^j$$

$g_j(x)$ is of degree jq, while the degree of the 'lowest' occuring term is p^i+j-1. This is seen as follows:

$q\equiv 1\,(\mathrm{mod}\,p^i)$, so write $q=bp^i+1$. As $l+p^2\mathbb{Z}$ generates $(\mathbb{Z}/p^2\mathbb{Z})^\times$, $(b,p)=1$. We now have that

$$(x+1)^q=(x+1)(x+1)^{bp^i}\equiv(x+1)(x^{p^i}+1)^b=1+x+bx^{p^i}+\text{higher terms}$$

Thus

$$g_j(x)=((x+1)^q-1)^j-x^j=(x+bx^{p^i}+\text{higer terms})^j-x^j=$$
$$jbx^{p^i+j-1}+\text{higher terms}$$

This shows that

$$(\psi^q - 1)(\beta_n) = 0 \quad \text{for} \quad n \le p^i - 1$$

while

$$(\psi^q - 1)(\beta_n) = (n + 1 - p^i) b\beta_{n+1-p^i} + \text{'higher terms'}$$

It is seen that each of the blocks A_n of (2.2) projects to a $(p-1)$-dimensional subspace of the block $A_{n-p^{i-1}}$ for for $n \ge p^{i-1}$.

Consider now the commutative diagram

$$\begin{array}{ccccccc} & & K_0(BU;\mathbb{Z}/p) & \xrightarrow{\psi^q - 1} & K_0(BU;\mathbb{Z}/p) & & \\ & & i_* \downarrow & & \downarrow i_* & & \\ 0 \to K_0(K(\mathbb{F}_q);\mathbb{Z}/p) \to & & K_0(K;\mathbb{Z}/p) & \xrightarrow{\psi^q - 1} & K_0(K;\mathbb{Z}/p) & & \to K_1(K(\mathbb{F}_q);\mathbb{Z}/p) \to 0 \end{array}$$

As $\{i_*(\beta_j)\}_{j\in\mathbb{N},(j,p)=1}$ is a basis for $K_0(K;\mathbb{Z}/p) \cong K_0(BU;\mathbb{Z}/p)/\mathrm{Ker}(i_*)$, we see that $\psi^q - 1$ is injective on the blocks $i_*(A_n)$ with $n \ge p^{i-1}$. This gives the statement about $K_0(K(\mathbb{F}_q);\mathbb{Z}/p)$.

Furthermore, $\psi^q - 1 : K_0(K;\mathbb{Z}/p) \to K_0(K;\mathbb{Z}/p)$ is surjective: Let

$$x = \sum_{\substack{n=1 \\ (n,p)=1}}^{N} a_n \cdot i_*(\beta_n) \in K_n(K;\mathbb{Z}/p)$$

We show inductively in *N* that $x \in \mathrm{Im}(\psi^q - 1)$. As $x - (\psi^q - 1)(b^{-1}N^{-1}a_n\beta_{N+p^i-1})$ is of lower degree than *x*, we get the inductive conclusion, proving the statement about $K_1(K(\mathbb{F}_q);\mathbb{Z}/p)$

3. The *K*-localization of $\Sigma^\infty \mathbb{C}P^\infty$ and of $Q(\mathbb{C}P^\infty{}_+)$

In this section we calculate the *K*-localizations of the suspension spectrum of the space $\mathbb{C}P^\infty$ and of the corresponding infinite loop space $Q(\mathbb{C}P^\infty)$. We work at an fixed, odd prime *p*.

<u>Definition 3.1</u>

Define the polynomials $\{f_n(x)\}_{n\in\mathbb{N}}$ in $\mathbb{Z}[x]$ inductively by

(1) $f_0(x) = 1$,

(2) $f_1(x) = x^p - 1$, and

(3) $f_{n+1}(x) = f_n(x^p) - p^n f_n(x)$ for $n > 1$.

Proposition 3.2

(1) $f_n(x)$ is a polynomial of degree p^n and leading coefficient 1. Only monomials of degree divisible by p occurs in $f_n(x)$.

(2) For $j = 0,1,2,...,n$ we have that $f_{n+1}{}^{(j)}(1) = 0$.

(3) $(x-1)^n$ divides $f_n(x)$.

(4) If $\psi^p : \mathbb{Z}[x] \to \mathbb{Z}[x]$ is the operation defined by

$$\psi^p(g(x)) = g(x^p)$$

then

$$f_{n+1}(x) = (\psi^p - p^n) \circ (\psi^p - p^{n-1}) \circ ... \circ (\psi^p - p) \circ (\psi^p - 1)(f_0(x))$$

Proof:

Note that (1) and (4) are obvious from the definitions, and (3) follows directly from (2).

In order to show (2), we differentiate the relation (3.1.3) j times. Inductively, we get

$$f_{n+1}{}^{(j)}(x) = p^j x^{j(p-1)} f_n{}^{(j)}(x^p) + \sum_{k=1}^{j-1} s_k(x) f_n{}^{(k)}(x^p) - p^n f_n{}^{(j)}(x)$$

where the $s_k(x)$'s are polynomials.

For $j < n$ the statement that $f_{n+1}{}^{(j)}(1) = 0$ follows from the corresponding statement about $f_n(x)$. For $j = n$ we see that only two parts of $f_{n+1}{}^{(n)}(1)$ doesn't vanish: From $f_n(x^p)$ we get a part $p^n x^{n(p-1)} f_n{}^{(n)}(x^p)$, and from $-p^n f_n(x)$ we get $-p^n f_n{}^{(n)}(x)$. But these cancel for $x = 1$.

QED

Definition 3.3

Define, for $m,n > 0$, the map

$$\Psi_m{}^{(n)} : \mathbb{C}P^n \to BU$$

as the composite

$$\mathbb{C}P^n \longrightarrow \mathbb{C}P^\infty \xrightarrow{f_m(H)} BU,$$

where $f_m(H) : \mathbb{C}P^\infty \to BU$ classifies the virtual bundle $f_m(H)$.

Proposition 3.4

For $m \geq n+1$ the map $\Psi_m{}^{(n)}$ is null homotopic.

Proof:

$\Psi_m{}^{(n)} \in [\mathbb{C}P^n, BU] = \bar{K}^0(\mathbb{C}P^n)$ corresponds to the bundle $f_m(H)$ over $\mathbb{C}P^n$. From (3.1.3) we have that $f_m(H) = (H-1)^m g_m(H) = \xi^m g_m(H)$, and as $\xi^{n+1} = 0$ in $\bar{K}^0(\mathbb{C}P^n)$, K°(CP"), $\Psi_m{}^{(n)}$ is the null map.

QED

Definition 3.5

Define the map $\Psi_m^{(n)} : \Sigma^\infty \mathbb{C}P^n \to K$ as the composite

$$\Sigma^\infty \mathbb{C}P^n \xrightarrow{\Sigma^\infty \Psi_m^{(n)}} \Sigma^\infty BU \xrightarrow{i} K$$

Define the map $\Phi_m^{(n)} : \Sigma^\infty \mathbb{C}P^n \to \prod_{i=0}^{m} K$ as the composite

$$\Sigma^\infty \mathbb{C}P^n \xrightarrow{\Delta} \prod_{i=0}^{m} \Sigma^\infty \mathbb{C}P^n \xrightarrow{\prod_{i=0}^{m} \Psi_i^{(n)}} \prod_{i=0}^{m} K$$

where Δ is the diagonal map.

Let A be an Abelian group. $S^0 A$ denotes the Moore-spectrum with

$$\pi_i(S^0 A) = 0,\ i < 0,\ H^0(S^0 A) = A \text{ and } H^j(S^0 A) = 0 \text{ for } j \neq 0.$$

If X is a spectrum, then we denote $X \wedge S^0 A$ by XA or by $X[A]$.

Definition 3.6

Let, for $m \geq 1$, $R_m : \prod_{i=0}^{m} K \to \prod_{j=1}^{m} K\mathbb{Q}$ be the map given by

$$R_m(x_0, ..., x_m) = (Dx_1 - D(\psi^p - 1)x_0, Dx_2 - D(\psi^p - p)x_1, ..., Dx_m - D(\psi^p - p^{m-1})x_{m-1})$$

Here, the short exact sequence $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces a cofiber sequence

$$\Sigma^{-1}(K\mathbb{Q}/\mathbb{Z}) \xrightarrow{C} K \xrightarrow{D} K\mathbb{Q}$$

and $\psi^p : K \to K\mathbb{Q}$ is the stable Adams' operation, [A74], p.99.

Proposition 3.7

The homotopy fibre of R_m, $Fib(R_m)$, is equivalent to the spectrum

$$F_m = K \times \prod_{i=1}^{m} \Sigma^{-1}(K\mathbb{Q}/\mathbb{Z})$$

Proof:

The map $S_m : F_m \to \prod_{i=0}^{m} K$ is given by

$$S_m(x_0, x_1, ..., x_m) = (x_0, C(x_1) + \sigma_1(x_0), C(x_2) + \sigma_2(x_0), ..., C(x_m) + \sigma_m(x_0))$$

with $\sigma_n = \prod_{r=0}^{n-1} (\psi^p - p^r)$. It is easily seen that $R_m \circ S_m$ is null homotopic, and we get a lift of S_m to $\overline{S}_m : F_m \to Fib(R_m)$. We want to show that $\overline{S}_m$ is a homotopy equivalence.

The cofiber sequence $\Sigma^{-1}(K\mathbb{Q}/\mathbb{Z}) \to K \to K\mathbb{Q}$ shows that

$$\pi_j(\Sigma^{-1}K\mathbb{Q}/\mathbb{Z}) = \begin{cases} \mathbb{Q}/\mathbb{Z} & , j \text{ odd} \\ 0 & , j \text{ even} \end{cases}$$

and

$$\pi_j(F_m) = \begin{cases} \mathbb{Z} & , j \text{ even} \\ (\mathbb{Q}/\mathbb{Z})^m & , j \text{ odd} \end{cases}$$

The cofiber sequence

$$Fib(F_m) \to \prod_{i=0}^{m} K \xrightarrow{R_m} \prod_{i=1}^{m} K\mathbb{Q}$$

gives the exact sequence

$$0 \to \pi_{2n}(Fib(R_m)) \to \mathbb{Z}^{m+1} \xrightarrow{(R_m)_*} \mathbb{Q}^m \to \pi_{2n-1}(Fib(R_m)) \to 0$$

As $\psi^p{}_* : \mathbb{Z} \cong \pi_{2n}(K) \to \pi_{2n}(K\mathbb{Q}) \cong \mathbb{Q}$ is multiplication with $p^n / p^n = 1$, we see that $(R_m)_* : \mathbb{Z}^{m+1} \to \mathbb{Q}^m$ is given by

$$(R_m)_*(x_0,\ldots,x_m) = (x_1, x_2 + (p-1)x_1, \ldots, x_m + (p^{m-1}-1)x_{m-1})$$

Hence

$$\pi_j(Fib(R_m)) = \begin{cases} \mathbb{Z} & , j \text{ even} \\ (\mathbb{Q}/\mathbb{Z})^m & , j \text{ odd} \end{cases}$$

Consider now the diagram

$$\begin{array}{ccccc} F_m & \xrightarrow{T_m} & \prod_{i=0}^{m} K & \xrightarrow{P_m} & \prod_{i=1}^{m} K\mathbb{Q} \\ \bar{S}_m \downarrow & \searrow S_m & U_m \downarrow & & \downarrow V_m \\ Fib(R_m) & \longrightarrow & \prod_{i=0}^{m} K & \xrightarrow{R_m} & \prod_{i=1}^{m} K\mathbb{Q} \end{array}$$

Here the maps T_m, P_m, U_m and V_m are described as follows:

$$T_m(x_0, x_1, \ldots, x_m) = (x_0, C(x_1), \ldots, C(x_m)),$$
$$P_m(x_0, x_1, \ldots, x_m) = (D(x_1), \ldots, D(x_m))$$
$$U_m(x_0, x_1, \ldots, x_m) = (x_0, x_1 + \sigma_1(x_0), x_2 + \sigma_2(x_0) \ldots, x_m + \sigma_m(x_0))$$
$$V_m(x_1, \ldots, x_m) = (x_1 - (\psi^p - 1)x_0, x_2 - (\psi^p - p)x_1, \ldots, x_m - (\psi^p - p^{m-1})x_{m-1})$$

lt is easily seen that $S_m = U_m \circ T_m$ and that $V_m \circ P_m = R_m \circ U_m$, and thus the diagram is commutative.

As U_m and V_m induce isomorphisms in homotopy, a 5-lemma argument shows that the lift $\bar{S}_m$ of S_m is a homotopy equivalence.

QED

Proposition 3.8

$K_*(F_m;\mathbb{Q}) \cong K_*(K;\mathbb{Q})$ and $K_*(F_m;\mathbb{Z}/p) \cong \bigoplus_{i=0}^{m} K_*(K;\mathbb{Z}/p)$.

Proof:

This is evident, as $K_*(\Sigma^{-1}K\mathbb{Q}/\mathbb{Z};\mathbb{Q}) = 0$ and $K_*(K\mathbb{Q};\mathbb{Z}/p) = 0$.

QED

Proposition 3.9

The composite $R_m \circ \Phi_m{}^{(n)} : \Sigma^\infty \mathbb{C}P^n \to \prod_{j=1}^{m} K\mathbb{Q}$ is null-homotopic.

Proof:

This follows from the definitions and from (3.1.3).

QED

Definition 3.10

Define the map $R : \bigvee_{i=0}^{\infty} K \longrightarrow \bigvee_{i=0}^{\infty} K\mathbb{Q}$ as the direct limit of the maps $R_m : \prod_{i=0}^{m} K \to \prod_{j=1}^{m} K\mathbb{Q}$ (It follows from (3.6) that the R_m's are compatible for varying m).

From (3.5) we see that the composite

$$\Sigma^\infty \mathbb{C}P^n \xrightarrow{\Phi_m{}^{(n)}} \prod_{i=0}^{m} K \xrightarrow{i} \prod_{i=0}^{m+1} K$$

where the map i is the inclusion of the first $(m+1)$'st factors, equals

$$\Phi_{m+1}{}^{(n)} : \Sigma^\infty \mathbb{C}P^n \to \prod_{i=0}^{m+1} K$$

We thus get a map $\Phi^{(n)} : \Sigma^\infty \mathbb{C}P^n \to \bigvee_{i=0}^{\infty} K$.

Again, (3.5) and (3.4) shows that the composite

$$\Sigma^\infty \mathbb{C}P^n \xrightarrow{\Sigma^\infty j} \Sigma^\infty \mathbb{C}P^{n+1} \xrightarrow{\Phi^{(n+1)}} \bigvee_{i=0}^{\infty} K$$

equals $\Phi^{(n)} : \Sigma^\infty \mathbb{C}P^n \to \bigvee_{i=0}^{\infty} K$, where $j : \mathbb{C}P^n \to \mathbb{C}P^{n+1}$ is the inclusion. By taking the limit over n, we obtain a map

$$\Phi : \Sigma^\infty \mathbb{C}P^\infty \to \bigvee_{i=0}^{\infty} K$$

From (3.9) we conclude that $R \circ \Phi$ is null-homotopic, and we get a lift $\Phi : \Sigma^\infty \mathbb{C}P^\infty \to F$, where $F \cong \lim\limits_{\to} F_m \cong K \vee \bigvee_{i=0}^{\infty} \Sigma^{-1} K\mathbb{Q}/\mathbb{Z}$ denotes the homotopy fibre of the map R.

Theorem 3.11

Φ induces an isomorphism in $K_*(-;\mathbb{Z}/p)$-theory:

$$\Phi_* : K_*(\Sigma^\infty \mathbb{C}P^\infty; \mathbb{Z}/p) \xrightarrow{\cong} K_*(F;\mathbb{Z}/p)$$

Proof:

As $K_1(-;\mathbb{Z}/p)$ of both spectra vanishes, Bott periodicity shows that it suffices to consider the induced map Φ_* in $K_0(-;\mathbb{Z}/p)$-theory.

We calculate the action of the n'th factor map $\Psi_n : \Sigma^\infty \mathbb{C}P^\infty \to K$. We have that

$$\Psi_n^*(\xi) = \Psi_n^*(H-1) = \Psi_n^*(H) - \Psi_n^*(1) = f_n(H) - f_n(1) = f_n(H)$$

As $p \equiv 0 \pmod p$, (3.1.3) shows that $f_{n+1}(x) \equiv f_n(x^p) \equiv x^{p^{n+1}} - 1 \pmod p$. Thus

$$\Psi_n^*(\xi) = f_n(H) \equiv H^{p^n} - 1 \equiv (H-1)^{p^n} = x^{p^n} = (\mu_{p^n})^*(\xi).$$

From this we conclude that

$$(\Psi_n)_* = i_* \circ (\mu_{p^n})_*$$

with μ_{p^n} from (1.6).

Now we show that Φ_* is injective. Assume that $x = \sum_{n=1}^{N} a_n \beta_n$ is contained in $\operatorname{Ker}\Phi_*$.

Then $\Psi_{0*}(x) = i_*(x) = 0$, so $x = \sum_{n=1}^{N_1} a_n^{(1)} \cdot \gamma_n^{(1)}$ with $N_1 \le N/p$.

Next, $\Psi_{1*}(x) = i_* \circ \mu_{p^n *}(x) = 0$, and so $x = \sum_{i=1}^{N_2} a_n^{(2)} \gamma_n^{(2)}$ with $N_2 \le N_1/p \le N/p^2$.

Repeating this argument, the injectivity follows.

In order to show that Φ_* is surjective, let $(y_0, y_1, ..., y_N, 0, 0, ...)$ be an element of $K_0(F;\mathbb{Z}/p) = \bigoplus_{i=0}^{\infty} K_0(K;\mathbb{Z}/p)$. Inductively we construct a sequence $x_0, x_1, ...$ of elements of $K_0(\Sigma^\infty \mathbb{C}P^\infty;\mathbb{Z}/p)$ such that $\Phi_*(x_i) = (y_0, y_1, ..., y_{i-1}, y_i, 0, 0, ...)$. As $y_i = 0$ for $j > N$, this process terminates after a finite number of steps, and we conclude that Φ_* is surjective.

First, $i_* = \psi_{0*}$ is surjective, so there exists $x_0^{(0)} \in K_0(\Sigma^\infty \mathbb{C}P^\infty;\mathbb{Z}/p)$ with $\Psi_{0*}(x_0^{(0)}) = y_0$. Write $x_0^{(0)} = \sum_{j=0}^{N} a_j \beta_j$

We adjust $x_0^{(0)}$ with elements from $\operatorname{Ker}\Psi_{0*} = span(\{\gamma_n^{(1)}\})$. Let $x_0^{(1)} = x_0^{(0)} + v$, where $v \in \operatorname{Ker}\Psi_{0*}$ satisfies the condition that $\Psi_{1*}(v) = -\Psi_{1*}(x_0^{(0)})$ – this is possible, as $\Psi_{1*} = i_* \circ \mu_{p*}|_{span\{\gamma_m^{(1)}\}}$ is surjective. Furthermore, as $x_0^{(0)}$ is of 'degree' N in the β_i's, v is of 'degree' at most N/p in the $\gamma_i^{(1)}$'s.

Inductively, we kill off the elements $\Psi_{m*}(x_0^{(m-1)})$ with linear combinations of the $\gamma_i^{(m)}$'s. Each adjustment is of 'degree' at most N/p^m, and so this process terminates after a finite number of steps. Thus, x_0 is defined to be $x_0^{(m)}$ for $m > \log_p(N)$.

Similarly, we can construct $x_1, x_2, ...$, and we conclude that Φ_* is surjective.

QED

Corollary 3.12

The map $\Phi : \Sigma^\infty \mathbb{C}P^\infty \to F$ is a $K_*(-;\hat{\mathbb{Z}}_p)$-equivalence.

Proof:

By using the Bockstein sequences in K-homology associated to the coefficient sequences

$$0 \to \mathbb{Z}/p^{n-1} \to \mathbb{Z}/p^n \to \mathbb{Z}/p \to 0$$

we inductively see that Φ is a $K_*(-;\mathbb{Z}/p^n)$-equivalence. By taking the limit, we obtain the result.

QED

We now turn to the rational type of $\Sigma^\infty \mathbb{C}P^\infty$. In [S72], it is shown that the map $i: Q(\mathbb{C}P^\infty) \to BU \times \mathbb{Z}$ (which Segal denotes by T), splits $Q(\mathbb{C}P^\infty)$ as $(BU \times \mathbb{Z}) \times C$, where the space C has finite homotopy groups. Translating this into a statement about spectra, we have

Proposition 3.13

The map $i: \Sigma^\infty \mathbb{C}P^\infty \to K < 0, \infty >= bu$ is a rational equivalence.

Definition 3.14

Define $r: K \to \prod_{i=-\infty}^{-1} \Sigma^{2i} S^0 \mathbb{Q}$ as follows: It is well known, that

$$K\mathbb{Q} \simeq \prod_{i=-\infty}^{\infty} \Sigma^{2i} S^0 \mathbb{Q}$$

We let r be the composite $K \xrightarrow{D} K\mathbb{Q} \xrightarrow{\pi} \prod_{i=-\infty}^{-1} \Sigma^{2i} S^0 \mathbb{Q}$, where we use the map D of (3.6), and where π is the projection onto the factors $\Sigma^{2i} S^0 \mathbb{Q}$ with $i \le -1$.

Let $\overline{bu}$ denote the homotopy fibre of r, and note that the natural map $bu \to K$, factors through $\overline{bu}$, as $\pi_i(bu) = 0$ for $i < 0$.

Proposition 3.15

The K-localization of bu is $\overline{bu}$.

Proof:

$\overline{bu}$ is obviously K-local, as both K and $\prod_{i=-\infty}^{-1} \Sigma^{2i} S^0 \mathbb{Q}$ are K-local. (Every rational spectrum is K-local, as it follows from the remark prcccding thm. (2.2) in [Mi]).

We have to show that the map $f: bu \to \overline{bu}$ is a K-equivalence. Let W denote the homotopy fibre of f. We need to show that $K_*(W) = 0$.

Let W_n denote the n-connected cover of W. We have the sequence of maps

$$0 = W_1 \to W_0 \to W_{-1} \to W_{-2} \to \ldots$$

and $W = \varinjlim_n W_{-n}$. As $K_*(W) = \varinjlim_n K_*(W_{-n})$, it suffices to show that $K_*(W_{-n}) = 0$.

This is done inductively: For $n < 0$, W_{-n} is the zero spectrum. We obtain W_{-n} from W_{-n+1} via the cofiber sequence

$$W_{-n+1} \to W_{-n} \to H(\pi_{-n}(W); -n)$$

where $H(\pi_{-n}(W); -n)$ is the Eilenberg-MacLane spectrum with the sole non-zero homotopy group

$$\pi_{-n}(H(\pi_{-n}(W); -n)) = \pi_{-n}(W)$$

$\pi_{-n}(W)$ is either zero or $\mathbb{Q}/\mathbb{Z}$, depending on whether *n* is even or odd. In both cases, $\pi_{-n}(W)$ is a torsion group, and $K_*(H(\pi_{-n}(W); -n)) = 0$, as it follows from [AH], thm. I. Inductively we see that $K_*(W_{-n}) = 0$.

QED

Definition 3.16

Let $\bar{F}$ be the homotopy fibre of the map

$$F = K \vee \bigvee_{i=1}^{\infty} \Sigma^{-1} K\mathbb{Q}/\mathbb{Z} \to \prod_{i=-\infty}^{-1} \Sigma^{2i} S^0 \mathbb{Q}$$

which is the map *r* of (3.14) on the first component, and zero on all the other components.

We have immediately that

$$\bar{F} = \overline{bu} \vee \bigvee_{i=1}^{\infty} \Sigma^{-1} K\mathbb{Q}/\mathbb{Z}$$

Theorem 3.17

The $K\mathbb{Z}_{(p)}$-localization of $\Sigma^\infty \mathbb{C}P^\infty$ is the spectrum $\bar{F} = \overline{bu} \vee \bigvee_{i=1}^{\infty} \Sigma^{-1} K\mathbb{Q}/\mathbb{Z}$

Proof:

This follows immediately from the rational statements (3.13) and (3.15), and from (3.12). Observe that the constituents of $\bar{F}$ are all *K*-local spectra.

QED

We wish to calculate the *K*-localization of the infinite loop space $Q(\mathbb{C}P^\infty)$. We use

Proposition 3.18 ([B82], (3.1))

Let *X* be a connective spectrum. Then there are natural isomorphisms

$$\pi_i(L_K \Omega^\infty X) \cong \pi_i(L_K X) \ , \ i > 2$$

$$\pi_i(L_K \Omega^\infty X) \cong \pi_i(X) \ , \ i < 2$$

and a natural short sequence:

$$0 \to \mathrm{tors}(\pi_2(L_K X)) \to \pi_2(L_K \Omega^\infty X) \to \pi_2(X)/\mathrm{tors}(\pi_2(X)) \to 0.$$

Theorem 3.19

The $K\mathbb{Z}_{(p)}$-localization of the space $Q(\mathbb{C}P^\infty)$ is

$$BU\times\mathbb{Z}\times\prod_{i=1}^{\infty}\Omega BU[\mathbb{Q}/\mathbb{Z}]<2,\infty>$$

where $\Omega BU[\mathbb{Q}/\mathbb{Z}]$ is the zero'th space in the Ω-spectrum $\Sigma^{-1}K\mathbb{Q}/\mathbb{Z}$.

Proof:

Making $\Sigma^\infty\mathbb{C}P^\infty$ into an Ω-spectrum and taking the corresponding infinite loop space map, we get $\Omega^\infty\Phi: Q(\mathbb{C}P^\infty)\to BU\times\mathbb{Z}\times\prod_{i=1}^{\infty}\Omega BU[\mathbb{Q}/\mathbb{Z}]$, and as the latter space is *K*-local, we get a map $L_K\Omega^\infty\Phi: L_KQ(\mathbb{C}P^\infty)\to BU\times\mathbb{Z}\times\prod_{i=1}^{\infty}\Omega BU[\mathbb{Q}/\mathbb{Z}]$.

(3.18) shows that this map is a homotopy equivalence in all dimensions except possibly 1 and 2.

We have that $\pi_2(\Sigma^\infty\mathbb{C}P^\infty)\cong\mathbb{Z}$, and that

$$\pi_2(L_K\Sigma^\infty\mathbb{C}P^\infty)=\pi_2(\overline{bu})\oplus\bigoplus_{i=1}^{\infty}\pi_2(\Sigma^{-1}K\mathbb{Q}/\mathbb{Z})\cong\mathbb{Z}\oplus\bigoplus_{i=1}^{\infty}0\cong\mathbb{Z}$$

(3.18) shows that $\pi_2(L_KQ(\mathbb{C}P^\infty))\cong\mathbb{Z}$, and we conclude that $L_K\Omega^\infty\Phi$ gives an equivalence in homotopy in dimension 2.

By killing off the π_1's of $\Omega BU[\mathbb{Q}/\mathbb{Z}]$ ($\pi_1(\Omega BU[\mathbb{Q}/\mathbb{Z}])\cong\mathbb{Q}/\mathbb{Z}$), we get the result.

QED

4. The *K*-localization of $\Sigma^\infty BG$ and $Q(BG_+)$

We now calculate the *K*-localization of the suspension spectrum of the space BG and of the corresponding infinite loop space $Q(BG_+)$, where $G=\mathbb{Z}/p^n$ is a cyclic *p*-group. *p* is all odd prime. As in §2 we select a prime *l*, such that $l+p^2\mathbb{Z}$ generates the unit group $(\mathbb{Z}/p^2\mathbb{Z})^\times$.

We start, by studying the spectrum $K(\mathbb{F}_l[G])$ – algebraic *K*-theory of the group ring $\mathbb{F}_l[G]$. This spectrum is defined by using an infinite loop space machine, e.g. [S74], on the category $\mathcal{GL}(\mathbb{F}_l[G])$ of all projective $\mathbb{F}_l[G]$-modules and $\mathbb{F}_l[G]$-isomorphisms; the group-law is the direct sum. This construction insures that $\pi_0(K(\mathbb{F}_l[G]))$ is isomorphic to $R_{\mathbb{F}_l}(G)$ – the Grothendieck group of projective $\mathbb{F}_l[G]$-modules.

Proposition 4.1

The spectrum $K(\mathbb{F}_l[G])$ splits as $\prod_{i=0}^{n}K(\mathbb{F}_{l_i})$, where $g=|G|=p^n$, $l_0=l$, and $l_i=l^{p^i}-l^{p^{i-1}}$ for $i>0$.

Proof:

The group ring $\mathbb{F}_l[G]$ is semi-simple, as $(l,p)=1$, and

$$\mathbb{F}_l[G]\cong\mathbb{F}_{l_0}\times\mathbb{F}_{l_1}\times\ldots\times\mathbb{F}_{l_n}$$

Indeed, the factor group $\mathbb{Z}/p^i$ of G has an irreducible representation over $\mathbb{F}_l$ of dimension p^i-p^{i-1}, as the finite field $\mathbb{F}_{l_i}$ has a p^i'th root of unity. This representation induces an irreducible representation V of G of the same dimension, giving the factor $\mathrm{Hom}_G(V,V)\cong\mathbb{F}_{l_i}$

QED

Corollary 4.2

$K_0(K(\mathbb{F}_l[G]);\mathbb{Z}/p)$ is a free $\mathbb{Z}/p$-module on p^n generators.

$K_1(K(\mathbb{F}_l[G]);\mathbb{Z}/p)=0$.

Proof:

This follows immediately from (4.1) and (2.7).

QED

Definition 4.3

Let G be a finite group. Define $\mathcal{T}_G$ to be the topological category, whose objects are the G-sets of the form $n(G/1), n\geq 0$, and whose morphisms are G-bijections. The topologies on the object set and on each morphism set are thle discrete topologies. We equip $\mathcal{T}_G$ with the composition $\coprod$ – disjoint union of sets.

The group completion $\Omega B(B\mathcal{T}_G)$ is an infinite loop space, cf. [S74], and in fact

Lemma 4.4

The infinite loop space $\Omega B(B\mathcal{T}_G)$ corresponding to $\mathcal{T}_G$ is $Q(BG_+)$.

Proof:

We have immediately that $\mathrm{Hom}_{\mathcal{T}_G}(n(G/1),n(G/1))\cong\Sigma_n\int G$ as a topological group. Thus, $\Omega B(B\mathcal{T}_G)\simeq\Omega B(\coprod_{n\geq 0}B(\Sigma_n\int G)$.

We have the 'Dyer-Lashof-equivalence' (cf. [MM], p.49):

$$Q(X)\simeq\Omega B((\coprod_{n\geq 0}E\Sigma_n\times_{\Sigma_n}X^n)/\approx),$$

where Σ_n acts on X^n by permuting the coordinates. The equivalence relation $\approx$ identifies points of $E\Sigma_{n-1}\times_{\Sigma_{n-1}}X^{n-1}$ with the subspace F_n of $E\Sigma_n\times_{\Sigma_n}X^n$ given by

$$F_n=\{(e;x_1,x_2,\ldots,x_n)\in E\Sigma_n\times_{\Sigma_n}X^n\mid\exists j:x_j=*\}.$$

Here * is the basepoint of X.

Since $E\Sigma_n\times_{\Sigma_n}(BG_+)^n=E\Sigma_n\times_{\Sigma_n}(BG)^n\coprod F_n$, and $E\Sigma_n\times_{\Sigma_n}(BG)^n\simeq B(\Sigma_n\int G)$, we get the result.

QED

The functor $\mathcal{T}_G \to \mathcal{GL}(\mathbb{F}_l[G])$ which sends the G-set $n(G/1)$ to its permutation representation $\mathbb{F}_l[G]^n$ is a map of permutative categories, so we get an infinite loop map $e: Q(BG_+) \to K(\mathbb{F}_l[G])$. We also denote by e the associated morphism between spectra

(4.5) $\quad e: \Sigma^\infty(BG_+) \to K(\mathbb{F}_l[G])$ e : W(BG,) K(IFt[GI).

We furthermore use the splitting of spectra

(4.6) $\quad \Sigma^\infty(BG_+) \cong \Sigma^\infty BG \vee S^0$,

where S^0 is the sphere spectrum, to define the map

(4.7) $\quad \bar{e}: \Sigma^\infty BG \to K(\mathbb{F}_l[G])$,

as the composite $\Sigma^\infty BG \longrightarrow \Sigma^\infty(BG_+) \xrightarrow{e} K(\mathbb{F}_l[G])$.

Theorem 4.8

$\bar{e}: \Sigma^\infty BG \to K(\mathbb{F}_l[G])$ gives an equivalence in $K_*(-;\mathbb{Z}/p)$-theory.

Proof:

From (1.8) and (4.2) we know that $K_*(\Sigma^\infty BG;\mathbb{Z}/p)$ and $K_*(K(\mathbb{F}_l[G]);\mathbb{Z}/p)$ are abstractly isomorphic. We construct a commutative diagram

(4.9)
$$\begin{array}{ccc} \Sigma^\infty BG & \xrightarrow{\bar{e}} & K(\mathbb{F}_l[G]) \\ \downarrow \Sigma^\infty B\chi & & \downarrow A \\ \Sigma^\infty \mathbb{C}P^\infty & \xrightarrow{\Phi} & L_K(\Sigma^\infty \mathbb{C}P^\infty) \end{array}$$

and show that the images if $\Phi \circ \Sigma^\infty B\chi$ and of A in $K_*(L_K(\Sigma^\infty \mathbb{C}P^\infty);\mathbb{Z}/p)$ are the same. As furthermore $\Phi \circ \Sigma^\infty B\chi$ and A give monomorphisms in $K_*(-;\mathbb{Z}/p)$-theory, we conclude that $\bar{e}$ is a $K_*(-;\mathbb{Z}/p)$-equivalence.

$\Sigma^\infty B\chi$ comes from the map $\chi: G \to S^1$ of (1.8), while Φ is defined in (3.10). A is the product of maps $A_j: K(\mathbb{F}_{l_j}) \to K$, where we observe the splittings (3.10) and (4.1). (As we work with $\mathbb{Z}/p$-coefficients, it doesn't matter whether we use K or $\Sigma^{-1}K\mathbb{Q}/\mathbb{Z}$, as $K \wedge S^0\mathbb{Z}/p \cong \Sigma^{-1}K\mathbb{Q}/\mathbb{Z} \wedge S^0\mathbb{Z}/p$). A_j is the 'Brauer lift' map of (2.6), and A_j goes into the $n-j$'th component of $L_K(\Sigma^\infty \mathbb{C}P^\infty) = \bigvee_{s=0}^{\infty} K$.

To show that (4.9) commutes, it suffices to show the commutativity of

(4.10)
$$\begin{array}{ccc} \Sigma^\infty BG & \xrightarrow{\bar{e}} K(\mathbb{F}_l[G]) \xrightarrow{\pi} & K(\mathbb{F}_l) \\ \downarrow \Sigma^\infty B\chi & & \downarrow A_j \\ \Sigma^\infty \mathbb{C}P^\infty & \xrightarrow{i \circ \mu_{p^{n-j}}} & L_K(\Sigma^\infty \mathbb{C}P^\infty) \end{array}$$

We thus have two elements $i \circ \mu_{p^{n-j}} \circ \Sigma^\infty B\chi$ and $A_j \circ \pi \circ \overline{e}$ of $[\Sigma^\infty BG, K]_* = K^*(BG)$. According to the Atiyah completion theorem, [A61], (7.2), $K^*(BG) = R(G)_p^\wedge$. Both $i \circ \mu_{p^{n-j}} \circ \Sigma^\infty B\chi$ and $A_j \circ \pi \circ \overline{e}$ corresponds to elements in $R(G)$: χ – the standard character of *G* sending $1 + p^n\mathbb{Z}$ to $\exp(2\pi i / p^n)$ – gives, after raising it to the p^{n-j}'th power, the character sending $1 + p^n\mathbb{Z}$ to $\exp(2\pi i / p^j)$. And the Brauer lift of the irreducible representation of *G* into $\mathbb{F}_{l_j}$ is easily seen to be the same character. We thus have established the commutativity of (4.9).

Now, the images of $i \circ \mu_{p^{n-j}} \circ \Sigma^\infty B\chi$ and of A_j in $K_*(-;\mathbb{Z}/p)$-theory are clearly the same, namely $i_*(<\beta_0, \beta_1, \ldots, \beta_{p^{n-j}}>)$.

QED

We are now able to calculate $L_K(\Sigma^\infty BG)$:

Definition 4.11

Let *q* be a prime power. Let $\mathcal{J}_q$ be the fibre of $\psi^q - 1 : K \to K$. Let $a : K(\mathbb{F}_q) \to \mathcal{J}_q$ be the map obtained from the diagram

$$\begin{array}{ccccc} K(\mathbb{F}_q) & \longrightarrow & K<0,\infty> & \xrightarrow{\psi^q - 1} & K<2,\infty> \\ {\scriptstyle a}\downarrow & & \downarrow & & \downarrow \\ \mathcal{J}_q & \longrightarrow & K & \xrightarrow{\psi^q - 1} & K \end{array}$$

Proposition 4.12

$a : K(\mathbb{F}_q) \to \mathcal{J}_q$ is a $K_*(-;\mathbb{Z}/p)$-equivalence.

Proof:

The proof is analogous to that of (3.15), the main point being that

$$K_*(H(\pi_{-n}(F);-);\mathbb{Z}/p) = K_*(H(\pi_{-n}(F) \otimes_{\mathbb{Z}} \mathbb{Z}/p; -n)) = 0$$

where *F* denotes the homotopy fibre of the map $a : K(\mathbb{F}_q) \to \mathcal{J}_q$.

QED

Proposition 4.13

Let *q* be a prime power. Let, as in [B79], p.269, $\overline{\mathcal{J}}_q$ be the homotopy fibre of the map $k : \mathcal{J}_q \to H(\mathbb{Q}, -1) = \Sigma^{-1}M\mathbb{Q}$, inducing the map $\mathbb{Z} \to \mathbb{Q}$ in $\pi_{-1}(-)$ (recall that the Hurewicz map $H : H_{-1}(-;\mathbb{Q}) \to \pi_{-1}(-;\mathbb{Q})$ is an isomorphism, as it follows from Serre theory). Then the K-localization of $K(\mathbb{F}_q)$ is $\overline{\mathcal{J}}_q$.

Proof:

As $\pi_{-1}(K(\mathbb{F}_q)) = 0$, we get a lift of the map *a* of (4.10) to $\overline{a} : K(\mathbb{F}_q) \to \overline{\mathcal{J}}_q$. (4.11) implies that $\overline{a}$ is a $K_*(-;\mathbb{Z}/p)$-equivalence.

The only non-zero $K_*(-;\mathbb{Q})$-homology groups of both $K(\mathbb{F}_q)$ and $\bar{\mathcal{J}}_q$ reside in dimension zero and are isomorphic to $\mathbb{Q}$. The map $\bar{a}$ is seen to be a $K_*(-;\mathbb{Q})$-equivalence, and the result follows.

QED

Definition 4.14

Define $\mathcal{K}(\mathbb{F}_l[G])$ as the *K*-local spectrum $\prod_{i=0}^{n}\mathcal{J}_{l_i}$. Define the *K*-local spectrum $\Sigma^{-1}\mathcal{K}(\mathbb{F}_l[G])[\mathbb{Q}/\mathbb{Z}]$ as the homotopy fibre of the rationalization map

$$\mathcal{K}(\mathbb{F}_l[G]) \to \mathcal{K}(\mathbb{F}_l[G])\mathbb{Q}$$

As a corollary to (4.12) we have

Corollary 4.15

The map $A = \prod_{i=0}^{n} a : K(\mathbb{F}_l[G]) = \prod_{i=0}^{n} K(\mathbb{F}_{l_i}) \to \prod_{i=0}^{n}\mathcal{J}_{l_i} = \mathcal{K}(\mathbb{F}_l[G])$ is a $K_*(-;\mathbb{Z}/p)$-equivalence.

Theorem 4.16

The *K*-localization of $\Sigma^\infty BG$ is $\Sigma^{-1}\mathcal{K}(\mathbb{F}_l[G])[\mathbb{Q}/\mathbb{Z}]$

Proof:

The composite

$$\Sigma^\infty BG \xrightarrow{\bar{e}} K(\mathbb{F}_l[G]) \xrightarrow{\mathcal{A}} \mathcal{K}(\mathbb{F}_l[G])$$

factors through $e' : \Sigma^\infty BG \to \Sigma^{-1}\mathcal{K}(\mathbb{F}_l[G])[\mathbb{Q}/\mathbb{Z}]$, as the homotopy groups of $\Sigma^\infty BG$ are finite.

lt follows from (4.8) and (4.15) that *e*' is a $K_*(-;\mathbb{Z}/p)$-equivalence, and as both $\Sigma^\infty BG$ and $\Sigma^{-1}\mathcal{K}(\mathbb{F}_l[G])[\mathbb{Q}/\mathbb{Z}]$ vanish rationally, the theorem follows.

QED

By using the splitting (4.6) and the fact that $L_K S^0 = \bar{\mathcal{J}}_1$, [H79], p.269, we get

Corollary 4.17

$$L_K(\Sigma^\infty BG_+) \simeq \bar{\mathcal{J}}_1 \vee \Sigma^{-1}\mathcal{K}(\mathbb{F}_l[G])(\mathbb{Q}/\mathbb{Z}) \simeq \bar{\mathcal{J}}_1 \vee \bigvee_{i=0}^{n}\Sigma^{-1}\mathcal{J}_{l_i}[\mathbb{Q}/\mathbb{Z}]$$

And from (3.18) we finally get:

Theorem 4.18

(1) The *K*-localization of the space $Q(BG)$ is $\Sigma K(\mathbb{F}_l[G])(\mathbb{Q}/\mathbb{Z})$ – the zero'th space of the Ω-Spectrum $\Sigma^{-1}\mathcal{K}(\mathbb{F}_l[G])[\mathbb{Q}/\mathbb{Z}]$.

(2) The *K*-localization of the space $Q(BG_+)$ is $K(\mathbb{F}_l)\times\Omega K(\mathbb{F}_l[G])(\mathbb{Q}/\mathbb{Z})$.

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