Oriented, equivariant *K*-theory and the Sullivan splittings

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The two other parts are

The *K*-localizations of Some Classifying Spaces

The Equivariant *K*-localization of the *G*-Sphere Spectrum

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The purpose of this paper is two-fold: To study oriented *G*-bundles and the corresponding *K*-theory, and to generalize the *p*-local splittings

$$F/O \simeq BSO \times \mathrm{Cok}\, J \quad \text{and} \quad SF \simeq J \times \mathrm{Cok}\, J$$

of Sullivan to the equivariant case.

This paper is divided into 5 parts. We start by recapitulating some essential facts about complex and real K_G-theory, and we study their classifying spaces.

In section 2 we introduce *G-SO*-bundles and *G-Spin*-bundles, and we find a connection between these and the 'equivariant Stiefel-Whitney-classes'.

In section 3 we study the space BSO_G in detail, at least when *G* is of odd order. Results about the λ-ring-structure on BSO_G of Atiyah-Tall and Atiyah-Segal are generalized – here it is necessary to assume that *G* is a *p*-group, where *p* is an odd prime, and that we are in the *p*-local situation.

In section 4 we study the space SF_G using the equivariant Adams' conjecture, and finally in section 5 we define the *e*-invariant and prove the Sullivan splittings.

Throughout the paper *G* is assumed to be finite. All *G*-Spaces are assumed to have a basepoint fixed under the *G*-action, and normally we consider only *G-CW*-complexes, which are finite and *G*-connected.

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1. Preliminary remarks about K_G- and KO_G-theory

In this section we briefly describe the functors $\bar{K}_G(-)$ and $\overline{KO}_G(-)$ and the corresponding classifying spaces.

$K_G(X)$ is defined in [S68], p.132, as the Grothendieck group of the additive semigroup of complex *G*-bundles over the *G-CW*-complex *X*. The tensor-product of *G*-bundles gives a multiplication on $K_G(X)$, and $K_G(X)$ becomes a commutative ring. Similarly, we have the ring $KO_G(X)$, obtained by using real rather than complex *G*-bundles.

$\bar{K}_G(X)$ is the reduced version of $K_G(X)$. It is defined as the subgroup of $K_G(X)$ generated by differences $E-F$ of complex *G*-bundles, such that for every $x \in X$, the fibres E_x and F_x over *x* are equivalent $\mathbb{C}G_x$-modules. Here $G_x = \{g \in G \mid gx = x\}$ is the isotropy group.

We define $\overline{KO}_G(X)$, the reduced version of $KO_G(X)$, in the same way.

Remark 1.1

If *X* is a *G*-connected *G-CW*-complex, i.e. for every subgroup *H* of *G* the fixed point space X^H is connected, then it follows from the local triviality condition of [L],

p. 258, that the difference $E - F$ of *G*-bundles over *X* is in $\bar{K}_G(X)$ if and only if the fibres E_* and F_* over the basepoint * are isomorphic *G*-modules.

By using an equivariant version of Brown's representation theorem, cf. [LMS], (1.5.11), we see that the functors $\bar{K}_G(-)$ and $\overline{KO}_G(-)$ are representable. We denote the classifying spaces by BU_G and BO_G, respectively.

Proposition 1.2

Let $U_1,...,U_m$ be a complete set of inequivalent, irreducible complex representations of *G*. Then

$$(BU_G)^G \simeq \prod_{i=1}^{m} BU$$

Proof:

From [S68], (2.2), we recall the isomorphism

$$(1.3) \qquad \mu : R(G) \otimes K(X) \to K_G(X)$$

where *X* is a trivial *G*-Space. As $\bar{K}_G(X) \cong [X, BU_G]^G \cong [X, BU_G{}^G]$, and $R(G)$ is a free $\mathbb{Z}$-module generated by $U_1,...,U_m$, the reduced version of this isomorphism

$$(1.4) \qquad \mu : R(G) \otimes \bar{K}(X) \to \bar{K}_G(X)$$

would imply the result.

μ maps $R(G) \otimes \bar{K}(X)$ into $\bar{K}_G(X)$: Let *E* and *F* be bundles over *X* with $E - F \in \bar{K}(X)$. Then the fibres E_x and F_x for every $x \in X$ have the same dimension. If *V* is a complex *G*-representation, then $\mu(V \otimes (E - F))$ is contained in $\bar{K}_G(X)$, as the fibres $V \otimes E_x$ and $V \otimes F_x$ over *x* are isomorphic $\mathbb{C}G$-modules.

On the other hand, $\mu(R(G) \otimes \bar{K}(X)) = \bar{K}_G(X)$: Let $\xi \in \bar{K}_G(X)$. In virtue of (1.3) we can find elements $\zeta_1, \ldots, \zeta_m$ in $K(X)$, such that $\xi = \sum_{i=1}^{m} U_i \otimes \zeta_i$. The fibre of the virtual *G*-bundle ξ over *x* is then, as an element of $R(G)$, given by

$$\xi_x = \sum_{i=0}^{m} U_i \otimes (\zeta_i)_x = \sum_{i=0}^{m} d_i \cdot U_i$$

where d_i is the complex dimension of $(\zeta_i)_x$. As $\xi \in \bar{K}_G(X)$, ξ_x vanishes as an element of $R(G)$, and we conclude that the d_i's are zero. Thus, the ζ_i's are contained in $\bar{K}(X)$, and (1.4) follows.

QED

In the real case we have the following:

Proposition 1.5

Let $U_1, U_2, \ldots, U_k$ be $\mathbb{R}G$-modules, $V_1, V_2, \ldots, V_m$ be $\mathbb{C}G$-modules, and $W_1, W_2, \ldots, W_n$ be $\mathbb{H}G$-modules, such that (2.6) in [K] is satisfied. Then

$$BO_G{}^G \simeq \prod_{x=1}^{k} BO \times \prod_{y=1}^{m} BU \times \prod_{z=1}^{n} BSp$$

Proof:

The proof is analogous to that of (1.2) and uses as input, the isomorphism

(1.6) $$\Phi : \bigoplus_{x=1}^{k} KO(X) \oplus \bigoplus_{y=1}^{m} K(X) \oplus \bigoplus_{z=1}^{n} KSp(X) \to KO_G(X)$$

of [K] (5.1). Here *X* is assumed to be a trivial *G*-space.

QED

Direct sum of vector-bundles makes $\bar{K}_G(-)$ and $\overline{KO}_G(-)$ into Abelian groups, and we thus get an 'additive' *G*-Hopf-space structures on BU_G and BO_G. (*G*-Hopf-spaces are defined in [Br], p.II.10.) We denote BU_G and BO_G with this 'additive' structure by $BU_G{}^{\oplus}$ and $BO_G{}^{\oplus}$.

It is also possible to define 'multiplicative' *G*-Hopf-structures on BU_G and BO_G. For a finite *G*-*CW*-complex *X* we consider the sets $1+\bar{K}_G(X)$ and $1+\overline{KO}_G(X)$. As every element in $\bar{K}_G(X)$ and $\overline{KO}_G(X)$ is nilpotent, cf. [S68], (5.1), the tensor-product makes $1+\bar{K}_G(X)$ and $1+\overline{KO}_G(X)$ into Abelian groups. By invoking Brown's representation theorem we get the representing *G*-Hopf-spaces $BU_G{}^{\otimes}$ and $BO_G{}^{\otimes}$.

The map $\bar{K}_G(X) \to 1+\bar{K}_G(X) : x \mapsto 1+x$ is a bijection for every *G*-*CW*-complex *X*, and it follows that $BU_G{}^{\oplus}$ and $BU_G{}^{\otimes}$ are *G*-homotopy-equivalent *G*-Spaces. Similarly we see that $BO_G{}^{\oplus}$ and $BO_G{}^{\otimes}$ are equivalent *G*-spaces.

For later use we need the following:

Proposition 1.7

Let *X* be a *G*-Space. If *E* is a complex *G*-bundle over *X*, then there exists a $\mathbb{C}G$-module *M* and a complex *G*-bundle $E^{\perp}$ such that $E \oplus E^{\perp} \cong \boldsymbol{M}$ (where $\boldsymbol{M}$ denotes the trivial *G*-bundle $M \times X \downarrow X$).

Similarly, if *F* is a real *G*-bundle, then there is an $\mathbb{R}G$-module *N* and a real *G*-bundle $F^{\perp}$ such that $F \oplus F^{\perp} \cong \boldsymbol{N}$.

Proof:

The complex case is (2.4) in [S68].

In the real case we do the following: $F \otimes_{\mathbb{R}} \mathbb{C}$ is a complex G-bundle, and we can thus find a complex G-bundle F_1, such that $(F \otimes_{\mathbb{R}} \mathbb{C}) \oplus F_1 \cong \boldsymbol{M}$, where M is a $\mathbb{C}G$-module. Now, F is a direct summand of the underlying real G-bundle $r(F \otimes_{\mathbb{R}} \mathbb{C})$ of $F \otimes_{\mathbb{R}} \mathbb{C}$ with orthogonal complement F_2. By taking underlying real G-bundles, we obtain the relation

$$F \oplus (r(F_1) \oplus F_2) \cong \boldsymbol{r(M)}.$$

Let $F^{\perp} = r(F_1) \oplus F_2$, and $N = r(M)$

QED

2. *G-SO*- and *G-Spin*-bundles

In this section we introduce G-SO-bundles and G-$Spin$-bundles, and we relate the classifying spaces of the functors $\overline{KSO}_G(-)$ and $\overline{KSpin}_G(-)$ to BO_G. We start by defining the G-spaces BSO_G and $BSpin_G$ as the G-1-connected and G-2-connected cover of BO_G, respectively:

Recall that if $n > 1$ and X is a $(n-1)$-connected space with $\pi_n(X)$ Abelian, then there is a map $k_n : X \to H(\pi_n(X), n)$, unique up to homotopy, such that $\pi_n(k_n)$ is the identity map on $\pi_n(X)$. Here $H(A,n)$ denotes the Eilenberg MacLane-space normally known as $K(A,n)$.

In the equivariant case we assume that X is a G-$(n-1)$-connected G-CW-complex, i.e. for every subgroup H of G we have that the fixed point space X^H is $(n-1)$-connected. We want to define a G-map $k_n : X \to H_G(\underline{\pi}_n(X), n)$, where $H_G(\underline{A}, n)$ is the equivariant Eilenberg-MacLane space classifying Bredon cohomology in dimension n with coefficients in the $\mathcal{O}_G$-group $\underline{A}$, cf. [El], p. 277. $\underline{\pi}_n(X)$ is the $\mathcal{O}_G$-group sending the orbit G/H to the Abelian group $\pi_n(X^H)$.

This map $k_n : X \to H_G(\underline{\pi}_n(X), n)$ is defined as the element of $[X, H_G(\underline{\pi}_n(X), N)]^G$ corresponding to $\underline{k}_n \in [\Phi X, \underline{H}(\underline{\pi}_n(X), n)]_{\mathcal{O}_G}$ under the bijection of [El], thm. 2. Here $\underline{k}_n : \Phi X \to \underline{H}(\underline{\pi}_n(X), n)$ is given by

$$\underline{k}_n(G/H) = k_n : \Phi X(G/H) = X^H \to H(\pi_n(X^H), n) = \underline{H}(\underline{\pi}_n(X), n)(G/H)$$

Definition 2.1

Let

$$w_1 : BO_G \to H_G(\underline{\pi}_1(BO_G), 1)$$

be the map k_1 from above. Let BSO_G denote the G-homotopy-fibre of w_1. (k_1 is well-defined, as BO_G is G-connected, and $\pi_1(BO_G{}^H)$ is Abelian, cf. (1.5)).

Similarly, let

$$w_2 : BSO_G \to H_G(\underline{\pi}_2(BSO_G) \otimes_{\mathbb{Z}} (\mathbb{Z}/2), 2)$$

be the map $r \circ k_2$, where $r : H_G(\underline{A}, n) \to H_G(\underline{A} \otimes_{\mathbb{Z}} (\mathbb{Z}/2), n)$ is the mod 2 reduction map, and where $k_2 : BSO_G \to H_G(\underline{\pi}_2(BSO_G), 2)$ is defined as above. Let $BSpin_G$ denote the G-homotopy-fibre of w_2. (An argument using the G-fibration

$$BSO_G \to BO_G \to H_G(\underline{\pi}_1(BO_G), 1)$$

shows that BSO_G is G-1-connected, and $k_2 : BSO_G \to H_G(\underline{\pi}_2(BSO_G), 2)$ is thus well-defined.)

Proposition 2.2

Let $U_1, U_2, ..., U_k$ be $\mathbb{R}G$-modules, $V_1, V_2, ..., V_m$ be $\mathbb{C}G$-modules, and $W_1, W_2, ..., W_n$ be $\mathbb{H}G$-modules, as in (1.5). Then

$$BSO_G{}^G \simeq \prod_{x=1}^{k} BSO \times \prod_{y=1}^{m} BU \times \prod_{z=1}^{n} BSp$$

and

$$BSpin_G{}^G \simeq \prod_{x=1}^{k} BSpin \times \prod_{y=1}^{m} BSpinU \times \prod_{z=1}^{n} BSp$$

where $BSpinU$ is the homotopy-fibre of the composite map

$$BU \xrightarrow{k_2} H(\pi_2(BU), 2) = H(\mathbb{Z}, 2) \xrightarrow{r} H(\mathbb{Z}/2, 2)$$

with r being the mod 2 reduction map.

Proof:

This follows immediately from (1.5) by taking the l-connected and 2-connected covers of $BO_G{}^G$. Recall that BSp is 2-connected, BU is 1-connected with $\pi_1(BU) \cong \mathbb{Z}$, and that $\pi_1(BO) \cong \mathbb{Z}/2$ and $\pi_2(BO) \cong \mathbb{Z}$.

QED

Remark 2.3

$BSpinU$ is not the same space as $BSpin^c$ of [St], p.292: We have that $\pi_n(BSpinU) \cong \pi_n(BU)$ for $n > 2$, and especially $\pi_6(BSpinU) \cong \mathbb{Z}$, while $BSpin^c$ sits in the fibration sequence

$$H(\mathbb{Z}, 2) \to BSpin^c \to BSO,$$

and therefore $\pi_6(BSpin^c) \cong \pi_6(BSO) = 0$.

From [L], p.257, we have the general definition of G-A-bundles, where A is the structure group. We explicify this definition in the cases where $A = SO(n)$ or $Spin(n)$:

Definition 2.4

A G-SO-bundle $E \downarrow X$ of dimension n is a G-map $p : E \to X$ between G-spaces such that

1) non-equivariantly, the map $p : E \to X$ is a $SO(n)$-bundle, and
2) for every $x \in X$ and $g \in G$ the restricted map $g|_{E_x} : E_x \to E_{gx}$ is a map of G_x-SO-modules.

If $E \downarrow X$ and $F \downarrow X$ are G-SO-bundles of the same dimension n, then a map $f : E \to F$ is a G-SO-bundle-map if f is both a G-map and a $SO(n)$-bundle-map.

It is easily seen that the pull-back f^*E along a G-map f again is a G-SO-bundle. Furthermore, the pull-backs along G-homotopic maps of the same G-SO-bundle are equivalent G-SO-bundles. We define the direct sum $E \oplus F$ of two G-SO-bundles $E \downarrow X$ and $F \downarrow X$ as $E \oplus F = \Delta^*(E \times F)$, where $\Delta : X \to X \times X$ is the diagonal map.

Finally, we get the Grothendieck-group $KSO_G(X)$ of isomorphism-classes of G-SO-bundles over the G-space X, and we define $\overline{KSO}_G(X)$ as the subgroup of $KSO_G(X)$ generated by differences of bundles $E - F$ satisfying

$$\forall x \in X : E_x \cong F_x \text{ as } G_x\text{-}SO\text{-modules.}$$

Definition 2.5

A G-Spin-bundle $E \downarrow X$ of dimension n is two G-spaces E and X and a G-map $p : E \to X$ such that

1) $p : E \to X$ is non-equivariantly a $Spin(n)$-bundle, and
2) for every $x \in X$ and $g \in G$ the restricted map $g|_{E_x} : E_x \to E_{gx}$ is a morphism of G-$Spin$-modules.

As with G-SO-bundles we get a Grothendieck-group $KSpin_G(X)$ and a reduced version $\overline{KSpin}_G(X)$.

Theorem 2.6

The classifying spaces of the functors $\overline{KSO}_G(-)$ and $\overline{KSpin}_G(-)$ are BSO_G and $BSpin_G$, respectively.

Proof:

We denote momentarily the classifying spaces for the functors $\overline{KSO}_G(-)$ and $\overline{KSpin}_G(-)$ by B_1 and B_2. We construct G-maps $\phi : B_1 \to BSO_G$ and $\psi : B_2 \to BSpin_G$ and show that they are G-homotopy-equivalences.

The spaces B_1 and B_2 are G-connected, as for every subgroup H of G, we have that

$$\pi_0(B_1^H) = \overline{KSO}_G(S^0 \wedge (G/H)_+) = 0$$

and

$$\pi_0(B_2^{\,H}) = \overline{KSpin}_G(S^0 \wedge (G/H)_+) = 0 .$$

We have a 'forgetful' map $\overline{\phi}: \overline{KSO}_G(X) \to \overline{KO}_G(X)$ for every *G*-connected *G*-*CW*-complex *X*, defined by sending a *G*-*SO*-bundle to its underlying orthogonal *G*-bundle. As $\overline{\phi}$ is a natural transformation between functors, we get a *G*-map $\overline{\phi}: B_1 \to BO_G$.

Let $E - F \in KSO_G(S^1)$, where *E* and *F* are *G*-*SO*-bundles. We decompose *E* (and *F*) according to [K], (4.1): Using the notation of (1.5), we can find real bundles $\eta_1, \eta_2, \ldots \eta_k$, complex bundles $\zeta_1, \zeta_2, \ldots \zeta_m$, and symplectic bundles $\xi_1, \xi_2, \ldots \xi_n$, such that

(2.7) $\qquad E = U_1 \otimes_{\mathbb{R}} \eta_1 \oplus \ldots \oplus V_1 \otimes_{\mathbb{C}} \zeta_1 \oplus \ldots \oplus W_n \otimes_{\mathbb{H}} \xi_n$

All the η_x's are *SO*-bundles, as the *SO*-action on *E* in $\eta_x = \mathrm{Hom}_{\mathbb{R}G}(U_x, E)$ gives a *SO*-action on η_x. Furthermore, our decomposition of *E* above is easily seen to be a decomposition of *G*-*SO*-bundles. Now, as $\overline{KSO}(S^1) = \overline{K}(S^1) = \overline{KSp}(S^1) = 0$, all *SO*-, *U*- and *Sp*-bundles over S^1 are trivial. Especially, the η_x's, the ζ_y's and the ξ_z's are trivial bundles, and *E* becomes a trivial *G*-bundle. We see that $\overline{KSO}_G(S^1) = 0$, and as $\overline{KSO}_G(S^1 \wedge (G/H)_+) \cong \overline{KSO}_H(S^1)$, we conclude that B_1 is *G*-1-connected.

The map $w_1 \circ \overline{\phi}: B_1 \to H_G(\underline{\pi}_1(BO_G),1)$ is null-homotopic, as $[B_1, H_G(\underline{\pi}_1(BO_G),1)]^G = H_G^{\,1}(B_1\,;\underline{\pi}_1(BO_G))$ is zero: B_1 is *G*-1-connected, and [Br], (11.7.1) shows that B_1 is *G*-homotopy-equivalent to a *G*-complex with no cells in dimensions less that 2. The definition of *G*-cohomology, [Br], (1.6.4), implies that $H_G^{\,1}(B_1\,;\underline{\pi}_1(BO_G))$ vanishes.

We get a lift $\phi: B_1 \to BSO_G$ of $\overline{\phi}$. We show that for every finite, *G*-connected *G*-*CW*-complex *X* the induced map $\phi: \overline{KSO}_G(X) \to [X, BSO_G]^G$ is an isomorphism. By using the equivariant Whitehead theorem and the fact that $\overline{KSO}_G(S^n \wedge (G/H)_+) = \overline{KSO}_H(S^n)$, it suffices to consider the case where $X = S^n$, $n \geq 1$. For $n = 1$, both $\overline{KSO}_G(S^1)$ and $[S^1, BSO_G]^G$ are zero.

Let *E* and *F* be *G*-bundles over S^n, $n > 1$, and let $E - F$ represent an element of $[S^n, BSO_G]^G = \overline{KO}_G(S^n)$. By using the decomposition (2.7), we get orthogonal bundles η_x over S^n. As $KO(S^n) = KSO(S^n)$, the η_x's are actually *SO*-bundles, and *E* becomes a *G*-*SO*-bundle (the complex and symplectic parts of *E* give no problem here). Thus, we see that ϕ is surjective.

To show that ϕ is injective, we show that the composite $\overline{\phi}$ is injective. So, let $E - F \in \mathrm{Ker}(\overline{\phi})$. Decompose *E* and *F* as above and note that we have *O*-isomorphisms between η_x and $\overline{\eta}_x$, *U*-isomorphisms between ζ_y and $\overline{\zeta}_y$, and *Sp*-isomorphisms between ξ_z and $\overline{\xi}_z$. But on S^0 there is no difference between *O*-isomorphisms and *SO*-isomorphisms of vector-bundles, as $\overline{KO}(S^n) \cong \overline{KSO}(S^n)$, and as *U*- and *Sp*-isomorphisms are *SO*-isomorphisms, we get *SO*-isomorphisms on all the components in the decompositions of *E* and *F*. These are assembled to show that $E \cong F$ as *G*-*SO*-bundles, and we see that $E - F = 0$, and $\overline{\phi}$ is injective. This shows that $B_1 = BSO_G$.

The part of the theorem concerning $\overline{KSpin}_G(-)$ and $BSpin_G$ is proved in the same way: The map $\overline{\psi} : B_2 \to BO_G$ is defined as the 'forgetful' map sending a *G*-Spin-bundle to its underlying orthogonal bundle. By using methods as above, we see that $w_1 \circ \overline{\psi}$ and $w_2 \circ \overline{\psi}$ are null-homotopic, and we get a *G*-map $\psi : B_2 \to BSpin_G$ – one of the main points is that if $E \downarrow X$ is a complex bundle, then the obstruction to *E* being a *Spin*-bundle is $w_2(E) \in H^2(X;\mathbb{Z}/2)$. But $w_2(E)$ is the image of $c_1(E)$ under the reduction map $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}/2)$. (It is this fact that makes the use of the space *BSpinU* necessary). By showing that the decomposition (2.7) respects *Spin*-structures, we see as before that ψ is a *G*-homotopy-equivalence.

QED

We remark that the *G*-spaces BSO_G and $BSpin_G$ are *G*-Hopf-spaces, cf. [Br], §11.4: The maps w_1 and w_2 are seen to be Hopf-maps by considering the functionality of the Elmendorfer construction – the map $k_n : X \to H(\pi_n(X), n)$ will in general be a Hopf-map when *X* is a Hopf-space. BSO_G and $BSpin_G$ with this Hopf-structure is denoted by $BSO_G^{\oplus}$ and $BSpin_G^{\oplus}$.

The tensorproduct of *G*-*SO*- and *G*-*Spin*-bundles gives the Hopf-spaces $BSO_G^{\otimes}$ and $BSpin_G^{\otimes}$ representing the functors $1 + \overline{KSO}_G(-)$ and $1 + \overline{KSpin}_G(-)$. As it is the case with BO_G, we have that $BSO_G^{\oplus}$ and $BSO_G^{\otimes}$, and that $BSpin_G^{\oplus}$ and $BSpin_G^{\otimes}$ are equivalent *G*-spaces, but the Hopf-space-structures will in general be different.

For later use we describe the rational type of BSO_G:

Lemma 2.8

Let *q* be a prime not dividing the order of the group *G*. Let *X* be a *G*-space, and let *Y* be a *q*-local infinite *G*-loop space. Then the *q*-local map

$$Fix : [X,Y]^G_{(q)} \to [\Phi X, \Phi Y]_{\mathcal{O}_G(q)}$$

sending the *G*-map $f : X \to Y$ to the $\mathcal{O}_G$-map

$$Fix(f) : G/H \mapsto (f^H : X^H \to Y^H)$$

is a bijection.

Proof:

This is essentially [LMS], (V.6.8) and (V.6.9): If $(|G|, q) = 1$, then

$$[X,Y]^G_{(q)} \cong \prod_{(H)} [X^H, Y^H]_{(q)}^{INV}$$

where the superscript 'INV' indicates that we are considering homotopy classes of 'invariant maps', [LMS] (V.6.5). But such an invariant homotopy class corresponds to a $\mathcal{O}_G$-homotopy class of $\mathcal{O}_G$-maps $\Phi X \to \Phi Y$.

QED

Proposition 2.9

Let $BSO_G\mathbb{Q}$ be the representing space of the functor $\overline{KSO}_G(-)\otimes\mathbb{Q}$. Then

$$BSO_G\mathbb{Q} \cong \prod_{n=2}^{\infty} H_G(\underline{\pi}_n(BSO_G)\otimes\mathbb{Q}, n)$$

Proof:

From (2.8) we have

$$\overline{KSO}_G(X)\otimes\mathbb{Q} \cong [X, BSO_G\mathbb{Q}]^G \cong [\Phi X, \Phi BSO_G\mathbb{Q}]_{\mathcal{O}_G}$$

For a subgroup *H* of *G* we have that

$$BSO_G\mathbb{Q}^H \simeq \prod_{x=1}^{k} BSO\mathbb{Q} \times \prod_{y=1}^{m} BU\mathbb{Q} \times \prod_{z=1}^{n} BSp\mathbb{Q}$$

as it follows from (2.2), and where $BSO\mathbb{Q}$, $BU\mathbb{Q}$ and $BSp\mathbb{Q}$ are the rational types of BSO, BU and BSp, respectively.

It is well-known that

$$BSO\mathbb{Q} \simeq \prod_{n=2}^{\infty} H(\pi_n(BSO)\otimes\mathbb{Q}, n)$$

and similarly for *BU* and *BSp*, and we see that

$$BSO_G\mathbb{Q}^H \cong \prod_{n=2}^{\infty} H_G(\pi_n(BSO_G{}^H)\otimes\mathbb{Q}, n)$$

By applying [El], thm. 2, we get the result.

QED

Of course, similar results holds for $BO_G\mathbb{Q}$, $BU_G\mathbb{Q}$, $BSp_G\mathbb{Q}$ and $BSpin_G\mathbb{Q}$.

3. The structure of BSO_G

In this section we study the structure of the space BSO_G via the λ-ring-structure on the functor $\overline{KSO}_G(-)$. The aim is to generalize results of Atiyah-Tall and Atiyah-Segal.

In the following we assume that *G* is a group of odd order. This implies that the numbers *k* and *n* of (1.5) are 1 and 0, respectively. Furthermore, $\underline{\pi}_1(BO_G)$ is the constant coefficient system $\mathbb{Z}/2$.

We start by showing an equivariant analogue of the splitting principle in Bredon cohomology, cf. [Hu], (16.5.2).

Lemma 3.1

Let $E \downarrow X$ be a *G*-bundle. Then there is a *G*-space $Q(E)$ and a *G*-map $q: Q(E) \to X$ such that $q^*(E)$ splits as a sum of *G*-line-bundles and the map

$$q^*: H_G(X; \underline{\pi}_1(BO_G)) \to H_G(Q(E); \underline{\pi}_1(BO_G))$$

is a monomorphism.

Proof:

As in the non-equivariant case, [Hu] (16.5.2), we construct $Q(E)$ inductively by going from *X* to $P(E)$ – the projective bundle of *E*. We see that the bundle $p^*(E)$ over $P(E)$ splits as a sum of a canonical line-bundle and another bundle of lower dimension than *E*, and we repeat this procedure on the latter bundle. (Here $p: P(E) \to X$ is the projection on the base space).

The injectivity of the map in Bredon-cohomology is also shown stepwise. It suffices to show that the map

$$p^*: H_G{}^*(X; \underline{\pi}_1(BO_G)) \to H_G{}^*(P(E); \underline{\pi}_1(BO_G))$$

is injective.

As the order of the group *G* is odd, and the coefficient system $\underline{\pi}_1(BO_G)$ is a $\mathbb{Z}_{(2)}$-module, we get from [LMS], (V.6.8) and (V.6.9), that there is a natural isomorphism

(3.2) $$\Phi: H_G{}^*(Z; \underline{\pi}_1(BO_G)) \to \bigoplus_{(H)} H^*(Z^H; \mathbb{Z}/2)$$

Here the sum is over all conjugacy classes of subgroups of *G*.

Using (3.2), we reduce the problem to show that

$$(q^H)^*: H^*(X^H; \mathbb{Z}/2) \to H^*(P(E)^H; \mathbb{Z}/2)$$

is injective for every subgroup *H* of *G*. But as *G* is of odd order $P(E)^H$ equals the projective bundle of the real bundle $E^H|_{X^H} \to X^H$, and we now use the non-equivariant splitting principle of [Hu], (16.5.2).

QED

If $E \downarrow X$ is a real *G*-bundle, we define $w_1(E) \in H_G{}^1(X; \underline{\pi}_1(BO_G))$ as $w_1(E - \boldsymbol{V})$, where $\boldsymbol{V}$ is the trivial bundle having $V = E_*$ as fibre. If $w_1(E) = 0$, we say that *E* is *G-orientable*.

Lemma 3.3

Let *E* and *F* be *G*-line-bundles over the *G*-connected *G*-space *X*. Then

$$w_1(E \oplus F) = w_1(E) + w_1(F).$$

Proof:

Let $L_G(X)$ be the semi-group of *G*-line-bundles over *X* with $\otimes$ as the composition. $L_G(-)$ is clearly a representable functor. Denote the classifying space by BL_G. Since $L_G(X)$ has a natural multiplication for all *X*, we see that BL_G is a *G*-Hopf-space.

We now get the following homotopy commutative diagram:

$$\begin{array}{ccc} BL_G & \xrightarrow{k_1} & H_G(\underline{\pi}_1(BL_G),1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ BO_G & \xrightarrow{w_1} & H_G(\underline{\pi}_1(BO_G),1) \end{array}$$

where the map *i* is induced by the map

$$L_G(X) \to \overline{KO}_G(X) : E \mapsto E - E_*$$

and *j* comes from the $\mathcal{O}_G$-group-homomorphism $\underline{\pi}_1(j) : \underline{\pi}_1(BL_G) \to \underline{\pi}_1(BO_G)$.

All these maps except possibly *i* are Hopf-maps. The commutativity of the diagram now gives the result.

QED

Corollary 3.4

Assume *X* is a *G*-connected *G*-space. Then $\overline{KSO}_G(X)$ is stable under the multiplication induced by $\otimes$.

Proof:

It suffices to show that if *E* and *F* are *G*-orientable then $E \otimes F$ is *G*-orientable, too. By using the splitting principle (3.1), we reduce to the case where *E* and *F* are line-bundles, and (3.3) gives the result.

QED

We recall that $KO_G(X)$ is a λ-ring: If *E* is a *G*-bundle over *X* and *n* a non-negative integer, then $\lambda^n E$ is the real *G*-bundle $\Lambda^n E$, where the *G*-action is given by

$$g(e_1 \wedge e_2 \wedge \ldots \wedge e_n) = (ge_1) \wedge (ge_2) \wedge \ldots \wedge (ge_n).$$

Proposition 3.5

Let *X* be a finite *G*-connected *G*-*CW*-complex. Then $KO_G(X)$ is a special, finite-dimensional λ-ring.

Proof:

$KO_G(X)$ is finite-dimensional, as every real *G*-bundle is finite-dimensional: Let *E* be a *G*-bundle over *X*, where *n* the dimension of a fibre of *E*. Then $\Lambda^m E = 0$ for $m > n$.

That $KO_G(X)$ is a special λ-ring follows from the splitting principle in KO_G-theory; see [tD], p.32.

QED

Corollary 3.6

$KSO_G(X)$ is a special λ-ring. $\overline{KSO}_G(X)$ is a λ-ideal in $KSO_G(X)$.

Proof:

We must show that if *E* is a *G*-oriented *G*-bundle, then $\Lambda^n E$ is *G*-oriented for all integers *n*. Using the splitting principle (3.1), we may assume that *E* is a sum of linebundles, $E = F_1 \oplus F_2 \oplus ... \oplus F_n$. We have the isomorphism

$$\Lambda^n(F_1 \oplus F_2 \oplus ... \oplus F_m) = \bigoplus (F_{i(1)} \otimes ... \otimes F_{i(n)})$$

where the sum is over all sequences $i(1) < i(2) < ... < i(n)$ of integers, cf. [Hu], (5.6.10). By using (3.3) we see that $w_1(\Lambda^n E)$ equals $\binom{m}{n} w_1(E) = 0$.

QED

Proposition 3.7

For *X* *G*-connected, the γ-ring $\overline{KSO_G}(X)$ is an oriented γ-ring.

Proof:

According to [AT], p.285 it suffices to show that for every $x \in \overline{KSO_G}(X)$ there exist *G*-bundles *E* and *F* over *X* such that $x = E - F$, and, if *n* denotes the dimension of *E* and *F*, then the linebundles $\Lambda^n E$ and $\Lambda^n F$ are the trivial one-dimensional *G*-bundle $V \times X \downarrow X$.

Write *x* as $E - \boldsymbol{V}$, where *E* is a *G*-bundle and ***V*** is the trivial bundle $V \times X \downarrow X$ for some *G*-module *V*, as in (1.7). Discarding the *G*-actions for a moment, we see that

$$0 = w_1(x) = w_1(E) - w_1(\boldsymbol{V}) = w_1(E)$$

and thus both $\Lambda^n E$ and $\Lambda^n F$ are trivial line-bundles, as $KSO(X)$ is an oriented λ-ring. We decompose $\Lambda^n E$ as in (1.5). As $\Lambda^n E$ is one-dimensional, this decomposition most be of the form $\Lambda^n E \cong \mathbb{R} \otimes_{\mathbb{R}} \eta_i$, as $\mathbb{R}$, the trivial one-dimensional representation, is the only 1-dimensional representation of *G*. If we ignore the *G*-action, $\mathbb{R}$ gives the trivial line-bundle, and $\Lambda^n E \cong \eta_i$ is a trivial bundle. Thus, both $\Lambda^n E$ and $\Lambda^n \boldsymbol{V}$ are isomorphic to $\mathbb{R}$.

QED

From now on we assume that *p* is an odd prime, and that *G* is a *p*-group.

Proposition 3.8

Let *X* a *G*-connected *G*-*CW*-complex. Then $\overline{KSO_G}(X) \otimes \hat{\mathbb{Z}}_p$ is a *p*-adic γ-ring.

Proof:

As *X* is *G*-connected, the natural inclusion $\overline{KSO_G}(X) \otimes \hat{\mathbb{Z}}_p \to \overline{KO_G}(X) \otimes \hat{\mathbb{Z}}_p$ is a monomorphism preserving the γ-ring-structure. [tD], (3.8.6) now gives the result.

QED

Theorem 3.9

There is a splitting of *G*-Hopf-spaces:

$$(BSO_G^{\oplus})_p^{\wedge} \simeq B_0^{\oplus} \times B_1^{\oplus} \times \ldots B_{m-1}^{\oplus} \quad , \quad m = \frac{p-1}{2}$$

Proof:

[AT], lemma 2.2, p.279 shows that, as $\overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p$ is a *p*-adic γ-ring, the domain of the Adams operations $\psi^k : \overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p \to \overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p$ in the variable *k* extends by continuity to operations

$$\psi^a : \overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p \to \overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p ,$$

where $a \in \hat{\mathbb{Z}}_p$.

Letting α be a generator of the finite factor $\mathbb{Z}/(p-1)$ of the splitting

$$(\hat{\mathbb{Z}}_p)^* \cong \mathbb{Z}/(p-1) \times \hat{\mathbb{Z}}_p$$

we have from [AT], p.284, that $\overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p$ splits canonically into eigenspaces for the operator ψ^α, the eigenvalues being α^i , $i = 0,1,\ldots, p-2$.

As this splitting is canonical in the space *X*, we get a corresponding splitting of the classifying space $(BSO_G)_p^{\wedge}$ into $p-1$ components.

Half of these components vanish: Let *i* be one of the odd numbers $1, 3, \ldots\ p-2$, and let $x \in \overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p$ be an eigenvector for ψ^α with eigenvalue α^i. As $\alpha^{(p-1)/2} = -1$, [AT], (5.2), p.264, shows that

$$\psi^{-1}(x) = \psi^{\alpha^{i(p-1)/2}}(x) = \alpha^{i(p-1)/2} x = -x$$

But as $\overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p$ is an oriented γ-ring, (3.7), ψ^{-1} acts as the identity operator, see [AT], p.285. Thus, *x* must be 0, and these all of 'odd' components vanish.

QED

Theorem 3.10

Let *p* be an odd prime. Then there is a splitting of *G*-Hopf-spaces

$$(BSO_G^{\otimes})_p^{\wedge} \simeq B_0^{\otimes} \times B_1^{\otimes} \times \ldots B_{m-1}^{\otimes} \quad , \quad m = \frac{p-1}{2}$$

Proof:

The proof is the same as that of (3.9) - the Adams operation ψ^a acts on $1 + \overline{KSO}_G(X)_p^{\wedge}$ by $\psi^a(x) = x^{a^i}$, where *x* is an element of the *i*'th eigenspace.

QED

Theorem 3.11

Let *p* be an odd prime and let *k* be an integer such that $k + p^2\mathbb{Z}$ generates the group of units in the ring $\mathbb{Z}/p^2\mathbb{Z}$. Then the cannibalistic class ρ^k induces an *G*-homotopy-equivalence of *G*-Hopf-spaces

$$\rho^k : B_0^{\oplus} \to B_0^{\otimes}$$

Proof:

[AT] (II 4.4).

QED

We obtain from [AS] thm.2 :

Theorem 3.12

Let p be a prime. Then there is a G-homotopy equivalence of G-Hopf-spaces

$$\delta : (BSO_G^{\oplus})_p^{\wedge} \to (BSO_G^{\otimes})_p^{\wedge}$$

A variant of δ is $\overline{\delta}$, which is the map ρ^k at the first component B_0 and δ at the rest of the components. As above we see that

(3.13) $\quad \overline{\delta} : (BSO_G^{\oplus})_p^{\wedge} \to (BSO_G^{\otimes})_p^{\wedge}$

is an equivalence of G-Hopf-spaces.

Remark 3.14

Actually, the results of [AT] and [AS] cannot be used directly in (3.9)-(3.13): In [AT] and [AS] it is assumed that we have a λ-ring R with an augmentation $\varepsilon : R \to \mathbb{Z}$, and then the results of [AT] holds for the augmentation ideal I.

We are in a more general situation, in that we have the λ-ring $KSO_G(X)$ and the λ-homomorphism $\varepsilon : KSO_G(X) \to RO(G)$ sending a G-bundle E to the representation E_*. The kernel of ε is $\overline{KSO}_G(X)$. It is possible to generalize the results of [AT] and [AS] to this case without any serious difficulties.

Counterexample 3.15

The crucial step in getting (3.9)-(3.12) from [AT] and [AS] is (3.8). When G is not a p-group, or when we do not localize at the order of the group, (3.8) does not hold. We give a simple counterexample:

If (3.8) did hold, then we would have, as in (1.5.6) in [AT], that the Adams' operation $\psi^k : \overline{KSO}_G(X) \to \overline{KSO}_G(X)$ would be p-adically continuous in the variable k.

Let $G = \mathbb{Z}/3$ be the cyclic group of order 3, and let p be the prime 5. Then $\overline{KSO}_G(S^{4n})$ is isomorphic to $RO(G)$ and is a free $\mathbb{Z}$-module of rank 2 with generators 1, V corresponding to the two irreducible $\mathbb{R}G$-modules of dimension 1 and 2, respectively. ψ^k maps $a1 + bV$ to $k^{2n}(a1 + bV)$ if $(k,3) = 1$ and to $k^{2n}(a + 2b)$ if $3 | k$.

If ψ^k was 5-adically continuous in *k*, then for every $x \in \overline{KSO}_G(S^{4n})$ and integer *m* we could find an integer *r*, such that for $5^r \mid s$ and integer *k*, we would have

$$\psi^{k+s}(x) - \psi^k(x) \in 5^m \cdot \overline{KSO}_G(S^{4n})$$

But if $3 \mid (k+s)$ and $(3,k)=1$ and $x = a1 + bV$, then

$$\psi^{k+s}(x) - \psi^k(x) = (((k+s)^{2n} - k^{2n})a + 2k^{2n}b)1 + k^{2n}V$$

which definitely not is contained even in $5 \cdot \overline{KSO}_G(S^{4n})$.

4. SF_G and the Adams' conjecture

We now proceed to study the *G*-space SF_G. Important ingredients in this analysis is the equivariant Adams' conjecture, due to McClure, cf. [MC], and the results of §3. Our standing assumption is that *p* is an odd prime, *G* is a *p*-group, and that all spaces are *p*-local.

Definition 4.1

Let $Q_G S^0$ be the *G*-loop-space $\lim_{\rightarrow} \Omega^V S^V$, where the limit is over all *G*-modules in a fixed, complete *G*-universe $\mathcal{U}$, cf. [LMS] p. 11. $Q_G S^0$ is a '*G*-ring-space', where the additive structure comes from the 'loop-sum' $* : \Omega^V S^V \to \Omega^V S^V$, which exists for every *G*-module *V*, and where the multiplication is composition of maps. We let the identity map be the basepoint of $Q_G S^0$.

Let SF_G be the *G*-connected cover of $Q_G S^0$. SF_G inherits a (multiplicative) *G*-Hopf-space structure from $Q_G S^0$.

Certain facts about $Q_G S^0$ are well-known - we recall from [S70], p.62, that

(4.2) $$(Q_G S^0)^G \simeq \prod_{(H)} Q(BW_H),$$

where the product is over all conjugacy classes (*H*) of subgroups of *G*. W_H is the Weyl-group $N_G(H)/H$. By taking connected covers we see that

(4.3) $$(SF_G)^G \simeq \prod_{(H)} Q_0(BW_H),$$

where $Q_0(BW_H)$ is the basepoint component of $Q(BW_H)$.

Definition 4.4

Let *X* be a finite *G*-*CW*-complex. The *G*-fibration $\xi : E \to X$ is a spherical *G*-*fibration* or a *G*-*sphere-bundle*, if

1) for every $x \in X$ there is a G_x-representation *V* such that the fibre E_x is G_x-homotopy-equivalent to S^V, and

2) the map $X \to E$ given by $x \mapsto$ (the basepoint of E_x) is a *G*-cofibration.

(This is the definition of [MC], p.230-231).

Fibre-wise smash-products makes the set of *G*-sphere-bundles over *X* into a semigroup, and the corresponding Grothendieck group is denoted $KF_G(X)$. The subgroup $\overline{KF}_G(X)$ is defined as follows

(4.5) $$E - F \in \overline{KF}_G(X) \iff \forall x \in X : E_x \simeq F_x \text{ as } G_x-\text{spaces}.$$

The functors $KF_G(-)$ and $\overline{KF}_G(-)$ are easily seen to be representable functors. We denote the classifying space of $\overline{KF}_G(-)$ by BF_G.

It follows from [W] that

(4.6) $$\underline{\pi}_0(BF_G) = 0 \quad \text{and} \quad \underline{\pi}_1(BF_G) \cong \underline{A}(G)^\times,$$

where the $\mathcal{O}_G$-group $\underline{A}(G)^\times$ is given by $\underline{A}(G)^\times(G/H) = A(H)^\times$ – the unit group of the Burnsidc ring $A(H)$. Furthermore, we see that BF_G is the classifying *G*-space of the *G*-monoid F_G – the subspace of $Q_G S^0$ consisting of *G*-homotopy-equivalences with the monoid structure coming from composition of maps.

Let BSF_G be the 1-connected cover of BF_G. It follows that BSF_G is the classifying space of the monoid SF_G, and thus

(4.7) $$\Omega BSF_G \simeq SF_G$$

Define the natural transformation $J_G : KO_G(X) \to KF_G(X)$ by sending the real *G*-bundle $E \downarrow X$ to its fibrewise one-point compactification $S^E \downarrow X$. It is immediately seen that J_G restricts to a natural transformation $\overline{KO}_G(X) \to \overline{KF}_G(X)$, and thus produces a *G*-Hopf-map $J_G : BO_G \to BF_G$. Furthermore, by killing off $\underline{\pi}_1$, we get a lift of $J_G : BSO_G \to BSF_G$.

Let F/O_G and SF/SO_G be the homotopy fibres of $J_G : BO_G \to BF_G$ and $J_G : BSO_G \to BSF_G$ respectively.

<u>Proposition 4.8</u>

The natural map $\theta : SF/SO_G \to F/O_G$ is a *G*-homotopy equivalence if *G* is of odd order or if we localize at an odd prime *p*.

<u>Proof:</u>

We have the *G*-homotopy commutative diagram:

$$\begin{array}{ccccc} SF/SO_G & \longrightarrow & BSO_G & \xrightarrow{J_G} & BSF_G \\ \theta\downarrow & & \downarrow & & \downarrow \\ F/O_G & \longrightarrow & BO_G & \xrightarrow{J_G} & BF_G \\ & & \downarrow & & \downarrow \\ & & H_G(\underline{\pi}_1(BO_G),1) & \xrightarrow{\psi} & H_G(\underline{\pi}_1(BF_G),1) \end{array}$$

Let *H* be a subgroup of *G*. $\pi_1(BO_G{}^H) \cong RO(H)/R(H)$ and $\pi_1(BF_G{}^H) = A(H)^\times$ are both 2-torsion groups, and θ is thus an equivalence away from 2.

If *G* is of odd order, then both $RO(H)/R(H)$ and $A(H)^\times$ are isomorphic to $\mathbb{Z}/2$. Furthermore, the non-zero element in $\overline{KO}_G(S^1 \wedge (G/H)_+) \cong \overline{KO}_H(S^1)$ is represented by the reduced Möbius-bundle with trivial *G*-action and, as in the non-equivariant case, is mapped by J_G to the non-trivial element in $\overline{KF}_G(S^1 \wedge (G/H)_+)$. Thus ψ is a *G*-homotopy equivalence and the result follows.

QED

The Adams conjecture relates J_G to the Adams-operations in *K*-theory. The non-equivariant version states:

Let *k* be an integer, $x \in KO(X)$. Then there exist an integer *n*, such that

$$k^n J(\psi^k x - x) = 0 \ .$$

By localizing at a prime *p*, satisfying $(p,k)=1$, we get rid of the factor *k*. Various attempts have been made to generalize the Adams conjecture to the equivariant case. In [FHM], theorem 0.4, it is shown that $k^n sJ(\psi^k x - x) = 0$, where $(k,|G|)=1$, and *s* is the minimal integer, such that $k^s \equiv \pm 1 \,(\mathrm{mod}\, |G|)$. The extra factor *s* is necessary – it insures that the 'fibres' of the virtual *G*-bundles $\psi^k x$ and *x* are the same element in $R(G_a)$ for every $a \in X$.

McClure has another variation, cf. [MC] (5.1). This uses a variant of the functor $KF_G(X)$:

Let *p* be a prime. Define the equivalence relation $\sim$ of stable *p*-equivalence on $KF_G(X)_{(p)}$ as follows: The *G*-sphere-bundles *E* and *F* are stably *p*-equivalent if there exists a real *G*-representation *V* and *G*-fiber maps

$$f_1 : S^V E \to S^V F \quad \text{and} \quad f_2 : S^V F \to S^V E$$

such that f_1 and f_2 have degrees prime to *p* on all fixed sets of each fibre.

Denote the set of stably *p*-equivalence classes in $KF_G(X)_{(p)}$ by $KF_G{}^{(p)}(X)$, and denote the reduced version by $\overline{KF}_G{}^{(p)}(X)$.

The relation between $KF_G(X)_{(p)}$ and $KF_G{}^{(p)}(X)$ is as follows, cf. [MC], (1.3):

Let X be a G-connected, finite G-CW-complex. Then there is a natural, short exact sequence

$$0 \to jO(G) \xrightarrow{\alpha} KF_G(X)_{(p)} \longrightarrow KF_G{}^{(p)}(X) \to 0$$

where $jO(G) = RO_0(G)/RO_h(G)$ ([tD] p.229), and α is the composite

$$jO(G) \rightarrowtail \mathrm{Im}(J : KO_G(*)_{(p)} \to KF_G(*)_{(p)}) \rightarrowtail KF_G(X)_{(p)}$$

Lemma 4.9

For X G-connected we have $\overline{KF}_G(X)_{(p)} \cong \overline{KF}_G{}^{(p)}(X)$.

Proof:

We have the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \overline{KF}_G(X)_{(p)} & \to & \overline{KF}_G{}^{(p)}(X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & jO(G)_{(p)} & \to & KF_G(X)_{(p)} & \to & KF_G{}^{(p)}(X) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & jO(G)_{(p)} & \to & KF_G(*)_{(p)} & \to & KF_G{}^{(p)}(*) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

QED

The equivariant Adams' conjecture [MC], (5.1) is now

Theorem 4.10

Let p be an odd prime and let k be an integer prime to p and $|G|$. Then the composite

$$(BSO_G)_{(p)} \xrightarrow{\psi^k - 1} (BSO_G)_{(p)} \xrightarrow{J} (BSF_G)_{(p)}$$

is null-homotopic.

Actually, this is not McClures formulation of the Adams conjecture, but upon using reduced KO_G - and KF_G -groups, and by using (4.9), we get the result above. The reason why this formulation doesn't involve extra factors is that we work in reduced KO_G - and KF_G -theory. This means that the condition that ξ and ξ^k have the same fibres over *x* in $R(G_x)$ for $x \in X$, is automatically fulfilled.

Corollary 4.11

There is a map $\alpha_k : (BSO_G)_{(p)} \to (F/O_G)_{(p)}$ such that

$$\begin{array}{ccccc} (F/O_G)_{(p)} & \longrightarrow & (BSO_G)_{(p)} & \xrightarrow{J} & (BSF_G)_{(p)} \\ & \nwarrow \alpha_k & \uparrow \psi^k - 1 & & \\ & & (BSO_G)_{(p)} & & \end{array}$$

commutes up to homotopy.

Definition 4.12

Let *G* be a group of odd order, and let *p* be an odd prime. Let *k* be an integer, such that $k + p^2\mathbb{Z}$ generates the unit group $(\mathbb{Z}/p^2\mathbb{Z})^\times$. Define the *G*-Spaces $J_G^{\oplus}$ and $J_G^{\otimes}$ as the homotopy fibres of the maps $\psi^k - 1 : BSO_G^{\oplus} \to BSO_G^{\oplus}$ and $\psi^k / 1 : BSO_G^{\otimes} \to BSO_G^{\otimes}$. As both $\psi^k - 1$ and $\psi^k / 1$ are Hopf-maps, $J_G^{\oplus}$ and $J_G^{\otimes}$ becomes *G*-Hopf-spaces. $J_G^{\oplus}$ and $J_G^{\otimes}$ are equivalent *G*-Spaces, but the Hopf-structures will in general be different.

Remark 4.13

In [FHM], (0.5) it is shown that J_G is the *G*-connected cover of equivariant, orthogonal, algebraic K-theory, $KO(\mathbb{F}_k, G)$, provided that *k* is a prime power.

5. The *e*-invariant and the Sullivan splittings

We now generalize the splittings

$$F/O \simeq BSO \times \mathrm{Cok}\, J \quad \text{and} \quad SF \simeq J \times \mathrm{Cok}\, J$$

of Sullivan to the equivariant case. We already have one of the maps needed to prove this, namely α_k , and we now define the other – the *e*-invariant.

As usual, *p* is an odd grime, *G* is a *p*-group, all spaces are *p*-local, and *k* is an integer such that $k + p^2\mathbb{Z}$ generates the unit group $(\mathbb{Z}/p^2\mathbb{Z})^\times$.

The main reason for studying *G*-*Spin*-bundles is that, as in the non-equivariant case, a G-$Spin(8n)$ -bundle has a Thom-class in KO_G -theory. Recall from [A], (6.1):

Theorem 5.1

Let Π be a compact Lie group, *V* a Π-*Spin*-module of dimension 8*n*, and *X* a compact *G*-Space. Then there is an element $u \in KO_G(X)$, defined by using the Dirac operator on *V*, such that multiplication with *u* induces an isomorphism

$$KO_G(X) \to KO_G(X \times V)$$

Theorem 5.2

Let *G* be a finite group, $E \downarrow X$ a $G\text{-}Spin(8n)$ -bundle over the compact *G*-connected *G*-*CW*-space *X*. Then there is an isomorphism

$$\Phi_E : KO_G(X) \to \overline{KO}_G(T(E))$$

where $T(E)$ is the Thom-complex of *E*.

Proof:

Let $R \downarrow X$ be the principal *G*-*Spin*(8*n*)-bundle corresponding to *E*, that is, we have a *G*-*Spin*(8*n*)-module *V* such that $E \cong R \times_{Spin(8n)} V$ (*V* is actually the fibre of *E*, at the base point of *X*, and the equivalence above follows from the fact that *X* is *G*-connected).

As *Spin*(8*n*) acts freely on *R*, we see that

$$KO_{G\times Spin(8n)}(R) \cong KO_G(R/Spin(8n)) \cong KO_G(X),$$

and that

$$KO_{G\times Spin(8n)}(R \times V) \cong KO_G(E) \cong \overline{KO}_G(T(E))$$

as *E* is not a compact *G*-space. The result follows now immediately from (5.1).

QED

We construct a *G*-Hopf-map $e : F/O_G \to BSO_G^{\otimes}$ as follows:

Let *X* be a finite *G*-connected *G*-*CW*-complex. Then the elements in $[X, F/O_G]^G$ can be described as 3-tuples (E, F, h), where *E* and *F* are stable *G*-bundles over *X*, such that $E - F \in \overline{KSO}_G(X)$ and where *h* is a fibrewise *G*-homotopy equivalence $h : S^E \to S^F$. (See [BM], p.146 for a closer description of the group structure on $[X, F/O_G]^G$.)

Since 2 is inverted, we can assume that that *E* and *F* are *G*-*Spin*-bundles, and by stabilizing, we can further assume that *E* and *F* are *G*-*Spin*(8*n*)-bundles.

Let $\Delta_E = \Phi_E(1) \in \overline{KO}_G(T(E))$ and $\Delta_F = \Phi_F(1) \in \overline{KO}_G(T(F))$ be the Thom-classes of *E* and *F*. *h* gives a map $T(E) \to T(F)$, and we define $e(E, F, h)$ as the unique element in $1 + \overline{KO}_G(X)$ satisfying

(5.3) $\qquad h^*(\Delta_F) = e(E, F, h) \cdot \Delta_E$

– observe that $\overline{KO}_G(T(E))$ is a free $KO_G(X)$-module of rank 1, and that Δ_E and $h^*(\Delta_F)$ are the image of units of $KO_G(X)$.

Proposition 5.4

We have a *G*-homotopy commutative diagram

$$\begin{array}{ccc} F/O_G & \xrightarrow{i} & BSO_G^{\oplus} \\ e\downarrow & & \downarrow \rho^k \\ BSO_G^{\otimes} & \xrightarrow{1/\psi^k} & BSO_G^{\otimes} \end{array}$$

where *k* is an integer, and $i : F/O_G \to BSO_G^{\oplus}$ is the 'inclusion' map.

Proof:

Let *X* be a finite, *G*-connected *G*-*CW*-complex, $(E,F,h) \in [X, F/O_G]^G$. Then

$$(1/\psi^k \circ e)(E,F,h) = (1/\psi^k)\left(\frac{h^*(\Delta_F)}{\Delta_E}\right) = \frac{\psi^k \Delta_E}{\Delta_E} \cdot \frac{h^*(\Delta_F)}{h^*(\psi^k \Delta_F)} =$$

$$\rho^k(E)\cdot(\rho^k(F))^{-1} = \rho^k(E-F) = \rho^k(i(E,F,h))$$

QED

Corollary 5.5

The composite $SF_G \longrightarrow F/O_G \xrightarrow{e} BSO_G^{\otimes}$ factors through J_G.

Proof:

We must show that the composite $SF_G \longrightarrow F/O_G \xrightarrow{e} BSO_G^{\otimes} \xrightarrow{1/\psi^k} BSO_G^{\otimes}$ is nullhomotopic. But from (5.4) we have the homotopy commutative diagram

$$\begin{array}{ccccc} SF_G & \xrightarrow{j} & F/O_G & \xrightarrow{i} & BSO_G^{\oplus} \\ & & e\downarrow & & \downarrow \rho^k \\ & & BSO_G^{\otimes} & \xrightarrow{1/\psi^k} & BSO_G^{\otimes} \end{array}$$

and as $i \circ j$ is null-homotopic, we get the result.

QED

Lemma 5.6

Let *k* be as in (4.12). Let $\alpha_k : BSO_G^{\oplus} \to F/O_G$ be the map of (4.11). Then the composite $e \circ \alpha_k : BSO_G^{\oplus} \to BSO_G^{\otimes}$ is *G*-homotopic to $\rho^k : BSO_G^{\oplus} \to BSO_G^{\otimes}$.

Proof:

We have the diagram

$$\begin{array}{ccc} BSO_G{}^{\oplus} & \xrightarrow{1-\psi^k} & BSO_G{}^{\oplus} \\ \alpha_k \downarrow & & \| \\ F/O_G & \xrightarrow{i} & BSO_G{}^{\oplus} \\ e \downarrow & & \downarrow \rho^k \\ BSO_G{}^{\otimes} & \xrightarrow{1/\psi^k} & BSO_G{}^{\otimes} \end{array}$$

which is homotopy commutative because of (4.11) and (5.4). As

$$\begin{array}{ccc} BSO_G{}^{\oplus} & \xrightarrow{1-\psi^k} & BSO_G{}^{\oplus} \\ \rho^k \downarrow & & \downarrow \rho^k \\ BSO_G{}^{\otimes} & \xrightarrow{1/\psi^k} & BSO_G{}^{\otimes} \end{array}$$

is commutative, too, we see that $1/\psi^k \circ (e \circ \alpha^k)$ and $1/\psi^k \circ \rho^k$ are *G*-homotopic maps.

As in [AII], p.152, it is possible to define ρ^k on a complex *G*-bundle $E \downarrow X$ by using the Thom-isomorphism $\Phi_E : K_G(X) \to \overline{K}_G(T(E))$, where $T(E)$ is the Thom-complex of *E*, cf. [A], (4.8). We have

(5.7) $\rho^k(E) = \Phi_E{}^{-1} \circ \psi^k \circ \Phi_E(1) \in K_G(X)$,

and from [AII], (5.4), we get

(5.8) $\Phi_E{}^{-1} \circ \psi^k \circ \Phi_E(x) = \rho^k(E) \cdot \psi^k(x)$, $x \in K_G(X)$

(This definition of ρ^k coincides with that of [AT], p. 281 and p. 268 – see [AT], p.286 ff.).

Letting $Y = S^{2n} = T(\mathbb{C}^n \downarrow *)$ and by using the exponential nature of ρ^k and its behaviour on complex line-bundles, we see that $\rho(\mathbb{C}^n \downarrow *) = k^n$ and from [tD], (3.5.1), and (5.8), we get

(5.9) $(\psi^k(\chi))(g) = k^n \cdot \chi(g)$, $g \in G$,

where $\chi \in \overline{K}_G(S^{2n})$ is considered as a complex character under the Thom-isomorphism

$$\Phi_{\mathbb{C}^n} : R(G) = K_G(*) \to \overline{K}_G(S^{2n})$$

As 2 is inverted, the map

$$\overline{KSO}_G(S^{2n}) \cong RO(G) \to R(G) \cong \overline{K}_G(S^{2n})$$

given by 'complexification' of representations, is injective, and preserves the λ-ringstructure.

Selecting a $\mathbb{Z}$-basis for $RO(G)$ consisting of the irreducible representations, we see that the matrix of the map $\psi^k - 1$ has non-vanishing determinant – modulo *k* this matrix is simply the diagonal matrix with –1 as the only entrys. We conclude that $\psi^k - 1$ induces monomorphisms

$$\pi_{2n}((\psi^k - 1)^H) : \pi_{2n}(BSO_G{}^H) \to \pi_{2n}(BSO_G{}^H)$$

for every subgroup H of G.

Going over to the multiplicative structure, we again have that $1/\psi^k$ gives monomorphisms in homotopy (for odd n $\pi_n(BSO_G{}^H)$ vanishes). We conclude that $e\circ\alpha_k$ and ρ^k give the same maps on the homotopy groups.

If we now consider ρ^k and $e\circ\alpha_k$ as natural transformations between the representable functors $\overline{KSO}_G(-)$ and $1+\overline{KSO}_G(-)$, we see that they coincide on the G-cells $S^n\wedge(G/H)_+$. We want to show that ρ^k and $e\circ\alpha_k$ coincide on every G-CW-complex.

As $KSO_G(BSO_G)$ is torsion-free, ([MR], at the bottom of p. 97,) it suffices to show that ρ^k and $e\circ\alpha_k$ coincide after rationalization. By applying (2.9), which states that both $BSO_G{}^{\oplus}\mathbb{Q}$ and $BSO_G{}^{\otimes}\mathbb{Q}$ are products of equivariant Eilenberg-MacLane-spaces, and Elmendorf's description of G-cohomology, [El], p.277, the problem reduces to show that for every integer $n>2$ and subgroup H of G the natural transformations

$$H^n(-;\pi_n(BSO_G{}^H)\otimes\mathbb{Q})\to H^n(-;\pi_n(BSO_G{}^H)\otimes\mathbb{Q})$$

induced by $\pi_n((\rho^k)^H)$ and $\pi_n((e\circ\alpha^k)^H)$ coincide. But $(\rho^k)^H$ and $(e\circ\alpha^k)^H$ agree on homotopy groups, and the result follows.

QED

Definition 5.10

Recall the G-Hopf-Space splitting

$$BSO_G{}^{\otimes}\simeq B_0{}^{\otimes}\times(B_0{}^{\otimes})^{\perp}$$

of (3.9), where $(B_0{}^{\otimes})^{\perp}\simeq B_1{}^{\otimes}\times\ldots\times B_{m-1}{}^{\otimes}$. Let π and $\pi^{\perp}$ be the projections

$$\pi: BSO_G\to B_0{}^{\otimes}\text{ and }\pi^{\perp}: BSO_G\to(B_0{}^{\otimes})^{\perp}.$$

Define $\beta: F/O_G\to BSO_G$ as the composite

$$F/O_G\xrightarrow{\Delta}F/O_G\times F/O_G\xrightarrow{e\times i}$$

$$BSO_G{}^{\otimes}\times BSO_G{}^{\oplus}\xrightarrow{Id\times\delta}BSO_G{}^{\otimes}\times BSO_G{}^{\otimes}\xrightarrow{\pi\times\pi^{\perp}}BSO_G{}^{\otimes}$$

Here Δ is the diagonal map, while δ is the map from (3.12).

Finally, define the G-space $\mathrm{Cok}\,J_G$ as the homotopy fibre of β.

We are now able to generalize the splittings of Sullivan [MN], (5.18)) to the equivariant case.

Theorem 5.11

β gives a splitting

$$F/O_G\simeq BSO_G\times\mathrm{Cok}\,J_G$$

Proof:

We show that $\beta \circ \alpha_k : BSO_G \to BSO_G$ is a G-homotopy equivalence:

$\beta \circ \alpha_k : BSO_G{}^{\oplus} \to BSO_G{}^{\otimes}$ G-homotopic to the composite

$$BSO_G{}^{\oplus} \xrightarrow{\Delta} BSO_G{}^{\oplus} \times BSO_G{}^{\oplus} \xrightarrow{\pi\rho^k \times \pi^{\perp}\delta(\psi^k - 1)} B_0{}^{\otimes} \times (B_0{}^{\otimes})^{\perp}$$

as it follows from (5.6) and (4.11). Separating $BSO_G{}^{\oplus}$ into $B_0{}^{\oplus}$ and $(B_0{}^{\oplus})^{\perp}$, we see that the composite

$$B_0{}^{\oplus} \to BSO_G{}^{\oplus} \xrightarrow{\beta\circ\alpha} B_0{}^{\otimes} \times (B_0{}^{\otimes})^{\perp}$$

equals

$$B_0{}^{\oplus} \xrightarrow{\Delta} B_0{}^{\oplus} \times B_0{}^{\oplus} \xrightarrow{\pi\rho^k \times 0} B_0{}^{\otimes} \times (B_0{}^{\otimes})^{\perp}$$

where 0 is a null-homotopic map, while the composite

$$(B_0{}^{\oplus})^{\perp} \to BSO_G{}^{\oplus} \xrightarrow{\beta\circ\alpha} B_0{}^{\otimes} \times (B_0{}^{\otimes})^{\perp}$$

becomes

$$(B_0{}^{\oplus})^{\perp} \xrightarrow{\Delta} (B_0{}^{\oplus})^{\perp} \times (B_0{}^{\oplus})^{\perp} \xrightarrow{\pi\rho^k \times \pi^{\perp}\delta(\psi^k - 1)} B_0{}^{\otimes} \times (B_0{}^{\otimes})^{\perp}$$

Thus, if we separate the homotopy groups of the spaces $BSO_G{}^{\oplus}$ and $BSO_G{}^{\otimes}$ into direct summands $\pi_n(BSO_G{}^{\oplus}) = \pi_n(B_0{}^{\oplus}) \oplus \pi_n((B_0{}^{\oplus})^{\perp})$ and $\pi_n(BSO_G{}^{\otimes}) = \pi_n(B_0{}^{\otimes}) \oplus \pi_n((B_0{}^{\otimes})^{\perp})$, the matrix of $\beta \circ \alpha_k$ becomes

$$\begin{pmatrix} \rho^k & \rho^k \\ 0 & \delta(\psi^k - 1) \end{pmatrix}$$

It suffices to show that $\rho^k : B_0{}^{\oplus} \to B_0{}^{\otimes}$ and $\delta(\psi^k - 1) : (B_0{}^{\oplus})^{\perp} \to (B_0{}^{\otimes})^{\perp}$ are *G*-homotopy-equivalences. The first fact follows from (3.11), while the second is more or less obvious – one needs the fact that δ preserves the splittings (3.9) and (3.10), but this follows from the construction of δ, (3.12) and [AS], thm. 3. Furthermore, on the factor $(B_0{}^{\oplus})^{\perp}$, the map $\psi^k - 1 : (B_0{}^{\oplus})^{\perp} \to (B_0{}^{\oplus})^{\perp}$ is a *G*-homotopy-equivalence, as this follows from the proof of (3.9), and the description of $(B_0{}^{\oplus})^{\perp}$ therein.

QED

Corollary 5.12

We have a splitting

$$SF_G \simeq J_G \times \mathrm{Cok}\, J_G$$

Proof:

We have the *G*-homotopy commutative diagram

$$\begin{array}{ccccc}
J_G{}^{\oplus} & \longrightarrow & BSO_G{}^{\oplus} & \xrightarrow{\psi^k - 1} & BSO_G{}^{\oplus} \\
{\scriptstyle\bar{\alpha}}\downarrow & & {\scriptstyle\alpha}\downarrow & & \| \\
SF_G & \longrightarrow & F/O_G & \longrightarrow & BSO_G{}^{\oplus} \\
{\scriptstyle\bar{\beta}}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\bar{\delta}}\downarrow \\
J_G{}^{\otimes} & \longrightarrow & BSO_G{}^{\otimes} & \xrightarrow{\psi^k / 1} & BSO_G{}^{\otimes}
\end{array}$$

where $\bar{\delta}$ is the snap from (3.13). Here the horizontal sequences are fibration sequences, and the maps $\bar{\alpha}$ and $\bar{\beta}$ are the maps induced by α and β.

Since $\beta \circ \alpha$ and $\overline{\delta}$ are *G*-homotopy equivalences, a five-lemma argument on every fixed point set diagram for every subgroup *H* of *G* shows that $\overline{\beta} \circ \overline{\alpha}$ is a *G*-homotopy equivalence. As $\overline{\delta}$ is a *G*-homotopy-equivalence, the homotopy fibres of β and $\overline{\beta}$ must be the same, namely $\operatorname{Cok} J_G$.

QED

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